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Tree Option Pricing Models

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Abstract

Discrete approximation of a continuous asset price process based on tree lattice models allows for intuitive and flexible valuation of financial derivatives. This note reviews binomial and trinomial recombining tree option pricing models. Our discussion includes various lattice specifications, estimation strategies for hedge sensitivities, and two-dimensional extensions. We also introduce examples of recently proposed techniques for computational efficiency and market consistency.

Contents

1 Introduction

There are fundamentally three numerical approaches to pricing of financial derivatives: partial differential equation (PDE), Monte Carlo, and tree model. Despite its numerical inefficiency the tree method is widely accepted since it is straightforward to implement and so flexible that it can be intuitively adapted to nonstandard options.

The tree model is a discrete approximation of a continuous process for the underlying instrument of a financial derivative. At a discrete time step the underlying asset price jumps by one of the predetermined amounts with respective transition probabilities. The first concrete realisation of the tree lattice approach to option valuation is a binomial tree model introduced by Cox, Ross and Rubinstein [CRR79] presenting an alternative route to the Black-Scholes formula for European vanilla options and a simple strategy to options with early exercise feature. In the binomial model the underlying asset price moves up or down at each time step. Improved valuation of options is expected with more possible states for the asset price evolution. Inserting a middle transition to the binomial model a trinomial tree was developed in [Boy86] that admits more flexibility in sake of computational simplicity. Over the years a wide variety of tree methods have been proposed in order to overcome difficulties in the primary models and refine numerical procedures for nonstandard option pricing.[1]

Using discrete time and asset price necessarily introduces approximation errors to resulting option value. In [FG99] the errors are classified into distribution error and nonlinearity error. Discrepancy between the distribution in the continuous model and approximated distribution in the lattice model for underlying asset price produces the distribution error while the two distributions have the same mean and variance. The nonlinearity error originates from nonlinear relation of the underlying asset price and option value, which is involved in payoff value function at expiry in general and on barriers for knock-in and knock-out options.

Although it is mathematically proved that tree lattice methods converge to the theoretical option value of a continuous-time model, the convergence is not smooth but oscillating due to the nonlinearity error [DD04]. Moreover, numerical simulation of the lattice model does not converge to the exact value because all of the numbers are truncated decimals in the numerical computation and each lattice model has a specific lower bound for the approximation error. These are the strong motivations for studying smooth and higher order convergence techniques [LR96, Jos10, Tia99, CP07].

In this note we review the binomial and trinomial tree option pricing models together with examples of lattice specifications, and enhancement of the computation techniques. Various types of tree lattice models have been studied from both of the theoretical and practical perspectives, and they are still in the revolutionary stage developing into a vast research area. Thus we must excuse that our references are not complete and readers should refer to the following literature and references therein. Gentle introduction to binomial and trinomial models and their implementation are given in [CS98] including application to exotic option pricing. Standard lattice specifications for the binomial tree are almost comprehensively explained in [Cha08]. For analytic and numerical approaches to vanilla and exotic options the complete guide [Hau06] summarises formulas and pesudo codes.

This note is organised as follows.

We start a discussion with binomial tree models in section 2 building a two-state model and

[1] Binomial and trinomial tree models are closely related to the PDE approach: time evolution of option price in the trees is equivalent to a finite difference method for solving the PDE (see section 3.1 for the trinomial model and [Jam03] for other cases).

extending it to multi-step cases, followed by definition of lattice specifications in 2.1, modelling of dividend payments in 2.2, approximation for nonconstant market parameters in 2.3, a two-dimensional generalisation in 2.4 and estimation of Greek values in 2.5. Then numerical results for the binomial tree models are summarised in 2.6.

Section 3 describes trinomial tree models. Our construction of the trinomial tree is based on risk-neutral probability measure. Subsequent discussions are on lattice specifications in 3.1, dividends of the underlying asset in 3.2, time-varying market parameters in 3.3, a two-dimensional generalisation in 3.4, estimation strategies for Greek values in 3.5, and numerical results in 3.6.

Section 4 is devoted to introducing alternative lattices and new techniques proposed to resolve issues identified in the earlier sections: flexible binomial tree in 4.1, adaptive mesh methods in 4.2, acceleration and truncation techniques in 4.3, and market-implied trinomial tree in 4.4.

All the figures are attached at the end of the paper.

2 Binomial Tree Models

In a binomial model underlying asset price follows a binomial process: at every moment the asset price changes to one of two possible prices during one step period.

We shall start with the simplest example by assuming an asset process S_t with $t = 0, \triangle t$ such that the asset price is S_0 at $t = 0$ and it goes up to uS_0 with probability p_u or down to dS_0 with probability p_d at $t = \triangle t$. The conservation of probability requires $p_u + p_d = 1$. Consider an European type option $C_t(S_t)$ which matures at $t = \triangle t$ and introduce the notation for payoff at maturity, $C_u = C_{\triangle t}(uS_0)$, $C_d = C_{\triangle t}(dS_0)$.

Let us construct a replicating portfolio of the option in terms of the underlying asset S_t and riskless bond B_t with the constant interest rate r, i.e., $B_t = e^{rt}$, and then, derive the current value of the option $C_0(S_0)$ from the absence of arbitrage opportunities. With δ notional of the underlying and n notional of the bond we have the conditions for replication at $t = \triangle t$,

$$\delta u S_0 + n e^{r\triangle t} = C_u \,, \qquad \delta d S_0 + n e^{r\triangle t} = C_d \,, \tag{2.1}$$

which are solved by

$$\delta = \frac{C_u - C_d}{(u-d)S_0} \,, \qquad n = \frac{uC_d - dC_u}{(u-d)e^{r\triangle t}} \,. \tag{2.2}$$

Absence of arbitrage requires that this portfolio and the option value should be also equal at $t = 0$. Thus we obtain

$$C_0 = \delta S_0 + n = e^{-r\triangle t}\left(\frac{e^{r\triangle t} - d}{u-d} C_u + \frac{u - e^{r\triangle t}}{u-d} C_d\right) \,. \tag{2.3}$$

If we introduce a new variable,

$$p = \frac{e^{r\triangle t} - d}{u-d} \,, \tag{2.4}$$

the current value of the option is rewritten as

$$C_0 = e^{-r\triangle t}\left(p\, C_u + (1-p) C_d\right) \,. \tag{2.5}$$

The coefficients p, $1-p$ can be, respectively, interpreted as the up probability p_u and the down probability p_d under a specific probability measure as far as $0 < p < 1$ is satisfied. From (2.4) this inequality is true if we impose

$$d < e^{r\triangle t} < u \,. \tag{2.6}$$

This condition is indeed the requirement for no arbitrage. Hence our interpretation for p, $1-p$ turns out to be appropriate.

In addition to that p in (2.4) has the properties of probability, it is actually derived from option valuation based on the risk-neutral probability: $p_u = p$ and $p_d = 1-p$ solve the equation, $\mathbb{E}(S_{\triangle t}|\mathcal{F}_0) = S_0 e^{r\triangle t}$, i.e.,

$$uS_0 p_d + dS_0 p_u = S_0 e^{r\triangle t}\,, \tag{2.7}$$

where $\mathcal{F}_0$ is the filtration at $t=0$. Therefore the current value of the claim C_0 is given by the expected value of the discounted payoff computed with the risk-neutral probability.

Hereafter our discussion is based on the risk-neutral probability measure meaning that the parameters of the binomial model should satisfy (2.4) in every time step at least *asymptotically* as $\triangle t \to 0$. The detailed discussion on the replicating portfolio strategy and risk-neutral pricing in the discrete model is given in [BR96].

So far we have discussed the two layer binomial model which has one time step and two final states. We shall consider its extension to a multi-step system. Let N to be the number of time intervals. For equally spaced intervals between current time 0 and expiry T we have the time period,

$$\triangle t = \frac{T}{N}\,. \tag{2.8}$$

Thus the lattice has $N+1$ layers separated by the time step $\triangle t$.

In a general binomial tree the number of nodes in the j-th layer becomes 2^j leading to unacceptable amount of computation in total. So we shall consider a recombining tree which can be constructed with the same set of u, d for all of the steps.[2] A state of the underlying asset is denoted as

$$S_{i,j} = S_0 u^j d^{j-i}\,, \qquad 0 \le i \le N\,, \quad 0 \le j \le i\,, \tag{2.9}$$

With the risk-neutral probability p the option price at the node (i,j) is given by the discounted expectation value,

$$C_{i,j} = e^{-r\triangle t}\left(p\, C_{i+1,j+1} + (1-p)\, C_{i+1,j}\right)\,. \tag{2.10}$$

Thus the option values in the i-th layer can be obtained once those in the $(i+1)$-th layer are known. The boundary values are given in the N-th layer: $C_{N,j}$ are the payoffs at expiry for the asset price $S_0 u^j d^{N-j}$. The formula (2.10) is modified depending on option types. For example, if the option has the early exercise feature, the option value $C_{i,j}$ is written as

$$C_{i,j} = \max\left\{e^{-r\triangle t}\left(p\, C_{i+1,j+1} + (1-p)\, C_{i+1,j}\right)\,,\ \text{(intrinsic value)}\right\}\,, \tag{2.11}$$

where the intrinsic value is the amount of cash obtained by the immediate exercise at the node. Another example of modification is a barrier option which we will discuss later.

In summary our option pricing strategy in the binomial model is as follows. We begin with the terminal values $C_{N,j}$ and compute the option values in the previous layer $C_{N-1,j}$ with the equation (2.10) (or (2.11)). Using the recursive formula at every node at ever time step we eventually obtain $C_{0,0}$ that is the current value of the option.

[2] If the value of the up and down factors is step-specific, that is, $u_i \neq u_j$, $d_i \neq d_j$ for $i \neq j$, the binomial tree is not recombining. Under certain situations, e.g., an underlying paying discrete dividends (sections 2.2 and 3.2) or time-varying volatility (sections 2.3 and 3.3), it is natural to consider the non-recombining property in order to capture the asset dynamics precisely. However, the non-recombinant tree requires the significantly increased amount of computation. For N time intervals, the non-recombining tree contains nodes of order $\mathcal{O}(2^{N+1})$ in contrast with $\mathcal{O}(N^2)$ for the recombining binomial tree. Therefore, our discussion in this note concentrates on the recombining tree and approximation methods are used for the cases mentioned above.

2.1 Lattice Specifications

In the previous section we have constructed the binomial model and explained the option pricing strategy with the risk-neutral probability distribution. The model parameters in the binomial tree are u, d and p which should satisfy the conditions (2.4) and (2.6) asymptotically. This section is devoted to discussing how to determine these parameters such that the binomial model approximates a continuous model. While the continuous model we consider in this note is limited to the Black-Scholes model where the underlying asset price is distributed log-normally, the binomial model can be incorporated with other distributions (see, e.g., [Rub98]).

In the risk-neutral measure the time evolution of risky asset is governed by the stochastic differential equation,

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t\,, \tag{2.12}$$

where W_t is the Brownian motion under the risk-neutral measure and r, σ are constants.

Since the property of the Black-Scholes model is captured by the mean and variance, we fix the binomial model parameters such that the same mean and variance are reproduced. Given a time interval $\triangle t$ we have the first and second moments in the Black-Scholes model,

$$\mathbb{E}\left(S_{\triangle t}\right) = e^{r\triangle t}S_0\,, \qquad \mathbb{E}\left(S_{\triangle t}^2\right) = e^{(2r+\sigma^2)\triangle t}S_0^2\,. \tag{2.13}$$

Matching of these two quantities between the Black-Scholes model and the binomial model requires the conditions,

$$puS_0 + (1-p)dS_0 = e^{r\triangle t}S_0\,, \qquad pu^2S_0^2 + (1-p)d^2S_0^2 = e^{(2r+\sigma^2)\triangle t}S_0^2\,. \tag{2.14}$$

As expected the first equation is exactly the same as the risk-neutrality condition (2.4). Thus the system is underdetermined and one more condition should be imposed to uniquely determine the binomial model parameters. While there are various choices for the extra condition, we shall introduce several useful examples in this note. The analysis across the lattice specifications is detailed in the paper [Cha08].

For notational simplicity we use a new variable,

$$\nu = r - \frac{1}{2}\sigma^2\,. \tag{2.15}$$

Below we only express p which is equal to the upper probability p_u in the risk-neutral measure. Since the probability conservation is exactly satisfied for all of the lattice specifications, the down probability is given by $1-p$.

Cox-Ross-Rubinstein

Imposing an additional constraint $ud = 1$ and solving the moment matching conditions (2.14), then the unique solution is given by

$$u = M + \sqrt{M^2-1}\,, \qquad d = M - \sqrt{M^2-1}\,, \qquad M = \frac{1}{2}\left(e^{(r+\sigma^2)\triangle t} + e^{-r\triangle t}\right)\,. \tag{2.16}$$

Recalling that our objective is to build a lattice model which converges to the Black-Scholes model in the $N \to \infty$ limit, it is not needed that subleading terms suppressed in the limit do

not exactly match between the two models for finite N. Modifying higher terms in $\triangle t$ leads to a simple form called Cox-Ross-Rubinstein (CRR) model [CRR79],

$$u = \exp\left(\sigma\sqrt{\triangle t}\right) , \quad d = \exp\left(-\sigma\sqrt{\triangle t}\right) , \quad p = \frac{1}{2} + \frac{\nu}{2\sigma}\sqrt{\triangle t} . \tag{2.17}$$

At this point it becomes clear what "asymptotically" means. The solution does not satisfy the conditions (2.14) exactly, but the equations hold only for its leading terms in $\triangle t$. In fact one can confirm that the distribution of the terminal underlying price in the CRR binomial process surely converges to the log-normal distribution (2.12). Furthermore it was demonstrated in [CRR79] that the European vanilla option price computed by the CRR model converges to the Black-Scholes option formula.

It should be noted that the probability p becomes negative for specific choices of r and σ. As discussed in [Hau04] the negativity of the intermediate probabilities does not necessarily imply the breakdown of the model. However, it can be a cause of instability issues in option valuation [Tri91].

Jarrow-Rudd

Jarrow and Rudd (JR) [JR83] (a similar model was proposed soon after CRR from a different point view [RJRB79]) proposed another lattice specification by imposing $p = 1/2$,

$$u = \exp\left(\nu\triangle t + \sigma\sqrt{\triangle t}\right) , \quad d = \exp\left(\nu\triangle t - \sigma\sqrt{\triangle t}\right) , \quad p = \frac{1}{2} . \tag{2.18}$$

Again this solves the equations (2.14) only approximately. The negative probability issue is not found in this lattice, but $u > d$ is not always true, especially for small number of time steps.

The asset price in the JR tree is growing on average with the constant rate. We call this a *growing tree* in contrast with the drift-free nature of the CRR model.

Tian

Tian considered the exact solution to the moment matching conditions [Tia93]. In addition to the first and second moment conditions (2.14), the condition for the third order moments is introduced. The third order moment in the Black-Scholes model is given by $S_0 e^{3(r+\sigma^2)\triangle t}$ which is equated to $p\,u^3 S_0 + (1-p)d^3 S_0$. Then the exact solution is

$$u = \frac{\hat{r}\hat{\sigma}}{2}\left(\hat{\sigma} + 1 + \sqrt{\hat{\sigma}^2 + 2\hat{\sigma} - 3}\right) , \quad d = \frac{\hat{r}\hat{\sigma}}{2}\left(\hat{\sigma} + 1 - \sqrt{\hat{\sigma}^2 + 2\hat{\sigma} - 3}\right) , \quad p = \frac{\hat{r} - d}{u - d} , \tag{2.19}$$

where

$$\hat{r} = \exp\left(r\triangle t\right) , \qquad \hat{\sigma} = \exp\left(\sigma^2\triangle t\right) . \tag{2.20}$$

Although CRR, JR and Tian models adapt the different probability measure for finite time step, all of them lead to the same result in the $N \to \infty$ limit.

Trigeorgis

It is convenient to impose the moment matching condition (2.14) in the log space by introducing $x = \log S_t$, where x is governed by the equation $dx = \nu dt + \sigma dW_t$. Expressing the up and down factors, u, d as $\exp(\triangle x_u)$, $\exp(\triangle x_d)$, then we have

$$p\triangle x_u + (1-p)\triangle x_d = \nu\triangle t , \qquad p\triangle x_u^2 + (1-p)\triangle x_d^2 = \sigma^2\triangle t + \nu^2\triangle t^2 . \tag{2.21}$$

One choice for the third condition in the log space is $\triangle x_u = \triangle x_d$. Then the solution is

$$u = \exp\left(\sqrt{\sigma^2 \triangle t + \nu^2 \triangle t^2}\right) , \quad d = \exp\left(-\sqrt{\sigma^2 \triangle t + \nu^2 \triangle t^2}\right) ,$$
$$p = \frac{1}{2} + \frac{\nu \triangle t}{2\sqrt{\sigma^2 \triangle t + \nu^2 \triangle t^2}} . \tag{2.22}$$

This lattice specification was developed by Trigeorgis [Tri91] in order to achieve a better accuracy and attain more stability than the CRR and JR models.

Jabbour-Kramin-Young

Another lattice specification based on the log space proposed by Jabbour, Kramin and Young (JKY) [JKY01] assumes that the tree is growing with the risk free rate. Recalling the the drift in the log space is ν instead of r we set $ud = e^{\nu \triangle t}$. Then the system (2.21) is exactly solved by

$$u = \exp\left(\nu \triangle t + \frac{(1-p)\sigma\sqrt{\triangle t}}{\sqrt{p(1-p)}}\right) , \quad d = \exp\left(\nu \triangle t - \frac{p\sigma\sqrt{\triangle t}}{\sqrt{p(1-p)}}\right) ,$$
$$p = \frac{1}{2} + \frac{\sigma\sqrt{\triangle t}}{2\sqrt{4 + \sigma^2 \triangle t}} . \tag{2.23}$$

Other common choices for the third condition in the system (2.21) include setting equal probabilities $p = 1/2$. However, there is no advantage in this choice in terms of the accuracy and rate of convergence [CS98].

Leisen-Reimer

When using any discrete model, there are two kinds of convergences to be taken into account: the convergence of binomial distribution to the geometric Brownian motion and that of option price computed in a discrete model to the solution of the continuous model. The relation between these two convergences is not obvious. It is not necessarily true that one lattice specification with higher order convergence to the risk-neutral distribution produces a better approximation to the option price.

In [LR96] authors proved that the order of convergence to the analytic price of the European option is equal to one for CRR, JR also and Tian. Thus, as far as valuation of the option is concerned, the efficiency of the computation is the same for all the three methods although the order of the convergence to the geometric Brownian motion is different.

Leisen and Reimer (LR) proposed a lattice specification with which the European option price exhibits the second order convergence [LR96]. The order of the convergence of the Leisen-Reimer model was conjectured in the original paper and proved later in [Jos10].[3]

The second order convergence is realised only for odd N by the following lattice specification,

$$u = \frac{\hat{r}\hat{p}}{p} , \qquad d = \frac{\hat{r} - p\,u}{1-p} , \qquad p = f\left(d_2\right) , \tag{2.24}$$

[3] In [Jos10] higher order convergence is achieved by extending the Leisen-Reimer lattice specification.

where $\hat{p} = f(d_1)$ and the function $f(z)$ is defined by

$$f(z) = \frac{1}{2} + \frac{\operatorname{sgn}(z)}{2}\left[1 - \exp\left(-\left(\frac{z}{N + \frac{1}{3} + \frac{0.1}{N+1}}\right)^2 \left(N + \frac{1}{6}\right)\right)\right]^{\frac{1}{2}},$$
$$d_1 = \frac{\log(S_0/K) + \nu T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}. \tag{2.25}$$

The definition of $\hat{r}$ is given in (2.20). Although the higher order convergence is demonstrated only for the European options, it is well known that this lattice specification tends to show a better convergence for other cases. One example for which the Leisen-Reimer lattice works very poorly is powered options whose payoff is $(\max\{S_T - K, 0\})^n$ for a call option and $(\max\{K - S_T, 0\})^n$ for a put option with $n > 0$.

Remarks on Convergence

We have seen that such a wide range of the lattice specifications is allowed in the binomial model. This flexibility originates from the fact that the constraints (2.14) are imposed only asymptotically, and one may consider that higher order terms in $\triangle t$ can be manipulated in arbitrary ways. From the mathematical point of view a natural question is "what is the minimum requirement for the model convergence to the Black-Scholes model in the large N limit?" In [Hsi83] the author presented the answer to this question and proved that any binomial model with the risk-neutral probability p such that $Np \to \infty$ as $N \to \infty$ converges to the Black-Scholes model. This condition is indeed weak since any probability with $0 < p < 1$ meets the requirement.

It is the rate of convergence to option prices rather than asset prices that we are more concerned with. In [DD04] the large N expansion of European vanilla option price in the CRR model is investigated and the explicit expansion formula is derived for the European call option price,

$$C_{\text{CRR}}(N) = \text{BS} + \frac{F(N)}{N} + o\left(\frac{1}{N}\right), \tag{2.26}$$

where BS is the Black-Scholes price and $F(N)$ is a bounded function that has no limit when N tends to infinity. This means that the $1/N^{1/2}$ term is absent, and $F(N)$ is responsible for the peculiar oscillatory behaviour (see section 2.6).

For American options, as their simple analytic formula is not available, the rate of the convergence is still uncovered. It is shown in [Lam98] that the valuation error for the American vanilla put option is bounded between 1/2 and 2/3, while numerical studies suggest that the convergence is order one [BD96, Lei96].

2.2 Dividend

In this section we consider options on an underlying asset which pays dividend yields. Before looking into the lattice model case we shall recall prescriptions frequently employed in the continuous model.

If the asset pays a continuous dividend at a fixed rate q, the Black-Scholes model (2.12) is modified as

$$\frac{dS_t}{S_t} = (r - q)\,dt + \sigma\,dW_t\,. \tag{2.27}$$

Thus the drift is reduced by the dividend rate and any derivative pricing strategies based on the Black-Scholes model are completed by $r \to r - q$.

What if the dividend payments are discrete? There are two cases we should consider: the dividends are proportional to the asset price at the dividend payment time, and each dividend is paid by cash. The former case can be handled within the framework of the Black-Scholes model. Let δ to be the known proportional dividend paid at the dividend date τ, i.e., the underlying asset price decreases by δS_τ. Rescaling the underlying asset,

$$\hat{S}_t = S_t(1-\delta)\,, \tag{2.28}$$

then we can assume that $\hat{S}$ follows the stochastic process (2.12). The option pricing procedure with the rescaled underlying price is exactly the same as the no dividend case.

If the discrete dividend is paid by cash at $t = \tau$ and the dividend amount is known as D, the underlying asset price jumps on the dividend date, $S_\tau \to S_\tau - D$. A simple prescription is to decompose the asset price,

$$S_t = \tilde{S}_t + D\, e^{-r(\tau - t)}\,, \quad t < \tau < T\,, \tag{2.29}$$

and find that $\tilde{S}_t$ follows the geometric Brownian motion (2.12). It should be noted that resulting option price on the cash dividend asset is scheme-dependent. Several approximation methods are compared in [Fri02] and a comprehensive review including American type options is, e.g., [NP13].

Below applications of the above techniques to the binomial tree are presented following the argument in [CS98].

Continuous Dividend

To achieve the convergence to the modified Black-Scholes model (2.27) we should shift the interest rate parameter in the binomial model,

$$r \to r - q\,. \tag{2.30}$$

Thus the lattice specifications in section 2.1 can be used with this shift and the the modification of equation (2.15), $\nu = r - q - \sigma^2/2$.

Discrete Proportional Dividend

The analogous approach to (2.28) is to assume that the dividend date is at $i\triangle t$ and express the value of the asset at node (i, j) as

$$\hat{S}_{i,j} = S_0\,(1-\delta)\,u^j d^{i-j}\,. \tag{2.31}$$

By using the rescaled asset price the binomial tree continues to be recombining even after the dividend payment. This argument can be extended for multi-dividend payment cases by rescaling the underlying asset price at every node when a dividend is paid.

Cash Dividend

If the underlying pays a cash dividend, the asset price is governed by a nonrecombining tree because the recombining condition, $(S_{i,j}u)d = (S_{i,j}d)u$, is indeed smeared,

$$(S_{i,j}u - D)d \neq (S_{i,j}d - D)u\,, \tag{2.32}$$

where the dividend time is between $(i+1)\triangle t$ and $(i+2)\triangle t$. However, using the same prescription as (2.29) allows one to use a recombining tree. Let us define

$$\tilde{S}_t = S_t - De^{-r(\tau-t)} \qquad \text{for} \quad t \leq \tau \,, \tag{2.33}$$

and assume that the volatility of $\tilde{S}_t$ is constant. Then the following asset price process produces a combining binomial tree,

$$S_{i,j} = \begin{cases} \tilde{S}_0 u^j d^{i-j} + De^{-r(\tau-t)} & \text{for} \quad t \leq \tau \\ \tilde{S}_0 u^j d^{i-j} & \text{for} \quad \tau < t \end{cases} \tag{2.34}$$

where $t = i\triangle t$. Although it is possible to extend this to multiple dividend payment cases, the recombining tree becomes an inaccurate approximation to the nonrecombining tree. Recently this issue is addressed by using interpolation technique [VN06].

2.3 Time-varying Volatility

Our discussion on the binomial tree approximation of the Black-Scholes model has assumed that the model parameters r, σ are constant. This is a too strong assumption because it is widely observed that volatilities in the market are not constant in time.[4] In this section we consider time-dependent interest rate and volatility, $r = r(t)$, $\sigma = \sigma(t)$, as a more general setup,

$$\frac{dS_t}{S_t} = r(t)dt + \sigma(t)\, dW_t \,. \tag{2.35}$$

Its lattice counterpart is actually a nonrecombining model because time-varying parameters in the continuous model naively reflect on the layer specific parameters in the binomial model, u_i, d_i, p_i and the recombining condition is broken, $u_i d_{i+1} S_{i,j} \neq d_i u_{i+1} S_{i,j}$. With the notation $\nu_i = r(i\triangle t) - \frac{1}{2}\left[\sigma(i\triangle t)\right]^2$ and $\sigma_i = \sigma(i\triangle t)$ the moment matching conditions in the log space (2.21) take the form,

$$p_i \triangle x_i - (1-p_i)\triangle x_i = \nu_i \triangle t_i \,, \quad p_i \triangle x_i^2 + (1-p_i)\triangle x_i^2 = \sigma_i^2 \triangle t_i + \nu_i^2 \triangle t_i^2 \,, \tag{2.36}$$

which are rewritten as

$$\begin{aligned} \triangle t_i &= \frac{1}{2\nu_i^2}\left(-\sigma_i^2 + \sqrt{\sigma_i^4 + 4\nu_i^2 \triangle x_i^2}\right) \,, \\ p_i &= \frac{1}{2} + \frac{\nu_i \triangle t_i}{2\triangle x_i} \,. \end{aligned} \tag{2.37}$$

However, if we use the following $\triangle x$ for all i,

$$\triangle x = \sqrt{\overline{\sigma}^2 \overline{\triangle t} + \overline{\nu}^2 \overline{\triangle t}^2} \,, \qquad \text{with} \;\; \overline{\sigma} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i \,, \;\; \overline{\nu} = \frac{1}{N}\sum_{i=1}^{N} \nu_i \,, \tag{2.38}$$

then $\overline{\triangle t}$ is approximately the average time step T/N [CS98].

[4] In addition the market implied volatility exhibit strike dependence $\sigma(K,t)$. Modelling this property is discussed in section 4.4.

We should note that the lattice specification above is not a good approximation in general. One can see this by computing the approximation error,

$$\triangle t_i - \overline{\triangle t} = \left(\frac{\overline{\sigma}^2}{\sigma_i^2} - 1 \right) \triangle t + \mathcal{O}\left(\triangle t^2\right) , \tag{2.39}$$

which shows that the error is the order of $\triangle t$ and the leading term is small only if $\sigma_i \simeq \overline{\sigma}$. Thus the recombining lattice approximation is not appropriate if $\sigma(t)$ changes rapidly in time.

Although a naive extension of the lattice specification to the trinomial model improves the computation, the same problem still exists and the continuous process is poorly approximated. Instead the time dependent feature of the model parameters can be nicely captured in the trinomial model by the implied tree (see section 4.4).

2.4 Two-dimensional Binomial Tree Model

For options whose payoff depends on multi-assets, one should consider multidimensional extension of the Black-Scholes model. Especially for the two-dimensional case, underlying asset prices are expressed in terms of correlated geometric Brownian motions,

$$dS_t^I = \left(r - q^I\right) S_t^I \, dt + \sigma^I S_t^I \, dW_t^I \, , \qquad I = 1, 2 \, , \tag{2.40}$$

where W_t^I are standard Brownian motion processes with a constant correlation ρ, i.e., $dW_t^1 dW_t^2 = \rho dt$. When constructing an extended binomial tree model converging to the two-dimensional Black-Scholes model, it is again convenient to introduce the log space for the asset prices, $x^I = \log S_t^I$. Then the equations (2.40) are rewritten as

$$dx^I = \nu^I dt + \sigma^I dW_t^I \, , \tag{2.41}$$

with $\nu^I = r - q^I - \frac{1}{2}(\sigma^I)^2$. A two-dimensional binomial model is specified by four probabilities, p_{uu}, p_{ud}, p_{du}, p_{dd} and jump sizes for the two assets, $\triangle x_u^I$, $\triangle x_d^I$, where for example p_{ud} represents the probability that the asset 1 goes up by $\triangle x_u^1$ and the asset 2 goes down by $\triangle x_d^2$ in one time step. Here we shall impose that up and down jump sizes are the same, i.e., $\triangle x^I \equiv \triangle x_u^I = \triangle x_d^I$ for each I. The first and second moments match if the following conditions are satisfied

$$\begin{aligned}
(p_{uu} + p_{ud}) \triangle x^1 - (p_{du} + p_{dd}) \triangle x^1 &= \nu^1 \triangle t \, , \\
(p_{uu} + p_{ud}) \left(\triangle x^1\right)^2 + (p_{du} + p_{dd}) \left(\triangle x^1\right)^2 &= \left(\nu^1\right)^2 \triangle t^2 + \left(\sigma^1\right)^2 \triangle t \, , \\
(p_{uu} + p_{du}) \triangle x^2 - (p_{ud} + p_{dd}) \triangle x^2 &= \nu^2 \triangle t \, , \\
(p_{uu} + p_{du}) \left(\triangle x^2\right)^2 + (p_{ud} + p_{dd}) \left(\triangle x^2\right)^2 &= \left(\nu^2\right)^2 \triangle t^2 + \left(\sigma^2\right)^2 \triangle t \, , \\
(p_{uu} - p_{ud} - p_{du} - p_{dd}) \triangle x^1 \triangle x^2 &= \rho \sigma^1 \sigma^2 \triangle t \, .
\end{aligned} \tag{2.42}$$

Together with the constraint on the probabilities, $p_{uu}+p_{ud}+p_{du}+p_{dd}=1$, the system is solved by

$$\begin{aligned}
&\triangle x^1=\sigma^1\sqrt{\triangle t}\,, \qquad \triangle x^2=\sigma^2\sqrt{\triangle t}\,,\\
&p_{uu}=\frac{\triangle x^1\triangle x^2+\left(\nu^1\triangle x^2+\nu^2\triangle x^1+\rho\sigma^1\sigma^2\right)\triangle t}{4\triangle x^1\triangle x^2}\,,\\
&p_{ud}=\frac{\triangle x^1\triangle x^2+\left(\nu^1\triangle x^2-\nu^2\triangle x^1-\rho\sigma^1\sigma^2\right)\triangle t}{4\triangle x^1\triangle x^2}\,,\\
&p_{du}=\frac{\triangle x^1\triangle x^2-\left(\nu^1\triangle x^2-\nu^2\triangle x^1+\rho\sigma^1\sigma^2\right)\triangle t}{4\triangle x^1\triangle x^2}\,,\\
&p_{dd}=\frac{\triangle x^1\triangle x^2-\left(\nu^1\triangle x^2+\nu^2\triangle x^1-\rho\sigma^1\sigma^2\right)\triangle t}{4\triangle x^1\triangle x^2}\,.
\end{aligned} \tag{2.43}$$

Again we modified higher order terms in $\triangle t$ to achieve the simple form of the solution.

Since each node in the two-dimensional tree is specified by the prices of the two underlying assets, we label the option values as $C_{i,j,k}$ where i represents the time layer with $0\le i\le N$ and the other two labels j and k respectively correspond to the asset 1 and asset 2 with $0\le j,k\le i$. Then the recursive formula which relate the option values in the $(i+1)$-th layer to those in the i-th layer is written as

$$C_{i,j,k}=e^{-r\triangle t}\left(p_{uu}C_{i+1,j+1,k+1}+p_{ud}C_{i+1,j+1,k}+p_{du}C_{i+1,j,k+1}+p_{dd}C_{i+1,j,k}\right)\,. \tag{2.44}$$

Therefore, once the option's payoff is computed at expiry, the current value of the option $C_{0,0,0}$ is eventually obtained by using this recursive formula at every time layer.

2.5 Hedge Sensitivities

In this section we consider Greek values in the one-dimensional and two-dimensional binomial tree models. The values of delta, gamma and theta can be estimated by using the option values at node points $C_{i,j}$. Since these quantities have been already computed when pricing the option, we obtain option price and the three Greek values once a single tree is built.

We shall first discuss the one-dimensional case. One simple approximation of the current value of delta is using the two nodes in the $t=\triangle t$ layer and taking finite difference ratio of the option values and asset prices,

$$\Delta=\frac{\partial C}{\partial S}\approx\frac{C_{1,1}-C_{1,0}}{S_{1,1}-S_{1,0}}\,. \tag{2.45}$$

To estimate the second order Greek one should step forward to the layer containing three nodes. Taking finite difference ratio twice gives

$$\Gamma=\frac{\partial^2 C}{\partial S^2}\approx\frac{\frac{C_{2,2}-C_{2,1}}{S_{2,2}-S_{2,1}}-\frac{C_{2,1}-C_{2,0}}{S_{2,1}-S_{2,0}}}{\frac{1}{2}\left(S_{2,2}-S_{0,0}\right)}\,. \tag{2.46}$$

The above computation of the delta and gamma can be regarded as estimates in the future times. The estimation at the current time is possible extending the tree up to $C_{-2,0}$ such that two extra nodes $C_{0,1}$, $C_{0,-1}$ appear at $t=0$.

In the binomial model only up node and down node are in the next layer $t = \triangle t$. Thus the option theta is estimated with the middle node in the layer $t = 2\triangle t$,

$$\Theta = \frac{\partial C}{\partial T} \approx \frac{C_{2,1} - C_{0,0}}{2\triangle t} \,. \tag{2.47}$$

However this not a good approximation if the tree is growing in time, i.e., $S_{2,1} \neq S_{0,0}$. In this case an alternative formula is suggested in [Rub94],

$$\Theta \approx rC_{0,0} - rS_{0,0}\Delta - \frac{1}{2}\sigma^2 S_{0,0}^2 \Gamma \,. \tag{2.48}$$

For example the theta value in the CRR model is computed by the former formula (2.47) whereas the JR model relies on the latter (2.48).

Other Greeks can be computed by repricing the option for small change in the corresponding parameter. For example, using the central finite difference approximation to vega yields

$$\frac{\partial C}{\partial \sigma} \approx \frac{C(\sigma + \Delta\sigma) - C(\sigma - \Delta\sigma)}{2\Delta\sigma} \,. \tag{2.49}$$

Note that computing one Greek value requires building two trees and the trees can not be reused for computation of other Greeks or the option price.

Greeks in the two-dimensional model can be estimated in a similar manner, but more ambiguity arises in the estimation. Also one needs to consider sensitivities to both of the two asset prices, so there are more varieties in the spot sensitivities.

Two kinds of option deltas are defined in the two-dimensional model: the sensitivity to the asset 1 and the sensitivity to asset 2. In the first time step both of the asset 1 price and asset 2 price change. Then it is in the next layer, $t = 2\triangle t$, that one is allowed to take the difference ration of the option values and asset prices with the other asset fixed. Another way to estimate the deltas is taking the average of the two difference ratios in the $t = \triangle t$ layer,

$$\begin{aligned} \Delta^1 &= \frac{\partial C}{\partial S^1} \approx \frac{1}{2}\left(\frac{C_{1,1,1} - C_{1,0,1}}{S_{1,1}^1 - S_{1,0}^1} + \frac{C_{1,1,0} - C_{1,0,0}}{S_{1,1}^1 - S_{1,0}^1}\right) , \\ \Delta^2 &= \frac{\partial C}{\partial S^2} \approx \frac{1}{2}\left(\frac{C_{1,1,1} - C_{1,1,0}}{S_{1,1}^2 - S_{1,0}^2} + \frac{C_{1,0,1} - C_{1,0,0}}{S_{1,1}^2 - S_{1,0}^2}\right) . \end{aligned} \tag{2.50}$$

For the second order Greeks we first take the average of the three difference ratios corresponding to the three states of the asset 2,

$$\begin{aligned} \Delta_{2,1}^1 &= \frac{1}{3}\left(\frac{C_{2,2,2} - C_{2,1,2}}{S_{2,2}^1 - S_{2,1}^1} + \frac{C_{2,2,1} - C_{2,1,1}}{S_{2,2}^1 - S_{2,1}^1} + \frac{C_{2,2,0} - C_{2,1,0}}{S_{2,2}^1 - S_{2,1}^1}\right) , \\ \Delta_{2,0}^1 &= \frac{1}{3}\left(\frac{C_{2,1,2} - C_{2,0,2}}{S_{2,1}^1 - S_{2,0}^1} + \frac{C_{2,1,1} - C_{2,0,1}}{S_{2,1}^1 - S_{2,0}^1} + \frac{C_{2,1,0} - C_{2,0,0}}{S_{2,1}^1 - S_{2,0}^1}\right) , \end{aligned} \tag{2.51}$$

then gamma for the asset 1 is estimated by

$$\Gamma^1 \approx \frac{\Delta_{2,1}^1 - \Delta_{2,0}^1}{\frac{1}{2}\left(S_{2,2}^1 - S_{2,0}^1\right)} \,. \tag{2.52}$$

Using the same prescription we can estimate gamma for the asset 2, Γ^2, and another second order Greek Γ^{12}, where the option price is differentiated by asset 1 and then by asset 2. Again this is not the unique way of estimating gamma. For example one can avoid the approximation based on taking average, c.f., (2.50), by computing the difference ration in the layer of $t = 3\triangle t$.

Since $u^I d^I = 1$ holds for $I = 1, 2$ in (2.43), the option theta is estimated in an analogous way to (2.47),

$$\Theta = \frac{C_{2,1,1} - C_{0,0,0}}{2\triangle t} \,. \tag{2.53}$$

As in the one-dimensional case, other Greek values are computed by the finite difference approximation.

We should note that the convergence of the Greek values is in general slower than that of the option price. Also the estimation using the nodes in adjacent layers results in a very poor approximation to the analytic formula in certain situations, for example, the case when the relevant nodes are separated by a boundary which causes the large nonlinearity error. While the central finite difference approximation (2.49) works better in these cases, the computation time is considerably increased since two distinct trees need to be built for evaluating each Greek value.

2.6 Numerical Results

In this section we present numerical results for the binomial tree models. All of the analytic formulas to be compared with our results are available in [Hau06].

The convergence behaviour of option prices and delta values is illustrated for all of the lattice specifications in Figures 1-10 where we show the normalised difference from the 'true value" defined by

$$\frac{V_{\text{Tree}} - V_{\text{True}}}{V_{\text{True}}}, \tag{2.54}$$

as a function of the number of time steps N for the range $(50, 249)$. V_{Tree} is a resulting value in a binomial model and V_{True} is a value in the continuous counterpart. If an analytic expression is not available, V_{True} is estimated by a tree model with sufficiently large number of time steps.

The convergence of an European vanilla call option price with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ is depicted for the CRR lattice, the JR lattice and the Tian lattice in Figure 1, for the Trigeorgis lattice and the JKY lattice in Figure 2, and for the LR lattice in Figure 3. The option delta for the same European option is shown in Figures 4, 5 and 6 for the same set of the lattice specifications. We have similar lessons from the option price and delta. In addition to the significant odd-even behaviour, it is observed that the error oscillates with a longer period causing that the greater approximation error can be produced even if we increase the number of time steps by 2. Since it produces much smaller errors and the number of time steps should be odd in the model, the LR lattice is separated in Figure 3 for the option price and in Figure 6 for the option delta. The figures suggest that the option price and delta in the LR lattice converge to those of the Black-Scholes model fast and smoothly.

It is not necessarily true that higher order convergence of European vanilla option price suggests that of nonstandard options. Especially, as pricing of path-dependent options in the binomial models introduces another source of the nonlinearity error, the LR lattice is not better than others any more. For example a down-and-out option with barrier H is priced by modifying

(2.10) as

$$C_{i,j} = \begin{cases} e^{-r\triangle t}\left(pC_{i+1,j+1} + (1-p)\,C_{i+1,j}\right) & \text{if } S_{i,j} > H \\ 0 & \text{if } S_{ij} \leq H \end{cases} \tag{2.55}$$

American knock-out options can be valued taking into account early exercise possibility in addition as in (2.11). If the option is European, knock-in type is priced through the in-out parity, whereas one should rely on alternative strategies for American knock-in options due to the absence of the parity (see e.g., [CS98]). We illustrate the case of an European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and the barrier $H = 25$ for the CRR lattice, the JR lattice and the Tian lattice in Figure 7, for the Trigeorgis lattice, the JKY lattice and the LR lattice in Figure 8. The severe oscillations are found for CRR and Trigeorgis, whereas the convergence is still oscillatory but moderately bounded for the JR, Tian and JKY cases resulting in a better approximation than the LR lattice. One may conclude that using a binomial model is not an optimal strategy for constant barrier options because shape of the barrier realised in the tree lattice is not flat but sawtooth, and the the effective barrier is located in a wrong place.

One of the most important applications of the binomial trees is to price American options. Since early exercise is never optimal for an American call option without dividend payments, we consider an American put option on an asset without dividend payments as the simplest example. The convergence behaviour is shown in Figure 9 for the CRR lattice, the JR lattice and the Tian lattice, and in Figure 10 for the Trigeorgis lattice, the JKY lattice and the LR lattice. Our parameter choice is $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$, and the "true value" in (2.54) is computed by the LR model with $N = 10001$. The figures are quite similar to the case of the European vanilla option: the odd-even property and long period oscillation are found. It is worth mentioning that the advantage in the LR lattice is limited for the present parameter choice in comparison with the outstanding rate of convergence for the European vanilla option.

An exchange-one-asset-for-another option is the right to exchange asset S^2 for asset S^1 at expiry. Then the payoff of the option is

$$\max\left(Q^1 S^1 - Q^2 S^2\right), \tag{2.56}$$

Q^1, Q^2 are quantities of the assets 1 and 2, respectively. We price the exchange-one-asset-for-another option with $S_0^1 = 29$, $S_0^2 = 31$, $T = 1$, $\sigma_0^1 = 0.3$, $\sigma_0^2 = 0.2$, $r = 0.1$, $\rho = 0.6$, $Q^1 = Q^2 = 1$ by using the two-dimensional lattice model in section 2.4. The convergence to the analytic formula is depicted in Figure 11. Besides the oscillatory behaviour as in the one-dimensional cases, the convergence is slow and the result is hardly improved even with a larger number of time steps. It is considered that this is because more approximation and numerical errors tend to accumulate in the two-dimensional model. Slightly better convergence is obtained if we use the trinomial model discussed in section 3.4.

3 Trinomial Tree Models

While the original argument on the binomial lattice method of CRR [CRR79] is based on the principle of option replication, the replication principle does not necessarily hold in other models. As shown for the binomial method, however, the option's fair value evaluated by the replication is the same as the price computed in the risk-neutral world, in which the fair value is expressed as the expected payoff under the risk-neutral probability discounted back to the valuation date

by the risk-free interest rate. This is not specific to the binomial model: any approximation procedure with a probability distribution that converges to the risk-neutral distribution in the continuous limit can be used to price options. Thus we shall start a discussion on trinomial tree models with the risk-neutral pricing.

In a trinomial tree an asset price changes to one of the three possible prices during one time step period meaning that given a set of three probabilities $\{p_u,\ p_m,\ p_d\}$ and three factors $\{u,\ m,\ d\}$, the three possible prices for the asset in the next time step are uS_0 with probability p_u, mS_0 with probability p_m or dS_0 with probability p_d. If we fix the time step $\triangle t$, the conditions for probability conservation and risk-neutral probability distribution are, respectively, written as

$$p_u + p_m + p_d = 1\,, \qquad p_u u S_0 + p_m m S_0 + p_d d S_0 = S_0 e^{r\triangle t}\,, \tag{3.1}$$

where r is the risk-free interest rate. Then the option price at $t = 0$ is expressed in terms of its terminal values at $t = \triangle t$, $C_u = C_{\triangle t}(uS_0)$, $C_m = C_{\triangle t}(mS_0)$ and $C_d = C_{\triangle t}(dS_0)$,

$$C_0 = e^{-r\triangle t}\left(p_u C_u + p_m C_m + p_d C_d\right)\,. \tag{3.2}$$

Towards the extension to a multi-step lattice we should note that the recombination is not automatic even if we use common $\{p_u,\ p_m,\ p_d\}$ for all the time steps. In fact one should impose

$$ud = m^2\,, \tag{3.3}$$

which ensures that the trinomial lattice is recombinant. Let $S_{i,j} = S_0 d^{i-j} m^j$ be the asset price at the node (i, j) with $0 \le i \le N,\ 0 \le j \le 2i$ and $C_{i,j}$ be its corresponding option value. Starting with payoffs at expiry $C_{N,j}$ the values of the option in the next layer are obtained by

$$C_{i,j} = e^{-r\triangle t}\left(p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j-1}\right)\,. \tag{3.4}$$

Using this recursive formula we eventually reach $C_{0,0}$ which is the current value of the option. The early exercise and other exotic features of options are captured in the same way as the binomial model.

3.1 Lattice Specifications

We consider a set of trinomial approximations to the geometric Brownian motion (2.12). Impose the matching of the mean and variance between the two models then we have the constraints,

$$p_u u S_0 + p_m m S_0 + p_d d S_0 = e^{r\triangle t} S_0\,, \qquad p_u u^2 S_0^2 + p_m m^2 S_0^2 + p_d d^2 S_0^2 = e^{(2r+\sigma^2)\triangle t} S_0^2\,. \tag{3.5}$$

Together with the conservation of the probability in (3.1) and the recombining condition (3.3), the system is underdetermined and has two degrees of freedom which we can make use of when building a trinomial tree. Also the moment matching conditions (3.5) are imposed only asymptotically as done for the binomial models, that leads to a wide variety of the lattice specifications. Here we shall introduce four examples of fixing the trinomial parameters. Some of other lattice specifications which are useful in specific contexts are described later in section 4.

Kamrad-Ritchken

Soon after the pioneer work on the trinomial model by Boyle [Boy86] Kamrad and Ritchken (KR) proposed a simpler and more efficient lattice specification [KR91]. The lattice parameters

are expressed in terms of a positive parameter λ,

$$\begin{gathered} u = e^{\lambda\sigma\sqrt{\triangle t}}\,, \qquad m = 1\,, \qquad d = e^{-\lambda\sigma\sqrt{\triangle t}}\,, \\ p_u = \frac{1}{2\lambda^2} + \frac{\nu}{2\lambda\sigma}\sqrt{\triangle t}\,, \quad p_m = 1 - \frac{1}{\lambda^2}\,, \quad p_d = \frac{1}{2\lambda^2} - \frac{\nu}{2\lambda\sigma}\sqrt{\triangle t}\,, \end{gathered} \tag{3.6}$$

If we set $\lambda = 1$, the possibility to take the middle brunch vanishes, then this trinomial lattice specification formally reduces to the CRR binomial model . By this reason the KR model is regarded as a trinomial extension of the CRR model. The authors also investigated the λ dependence in the option valuation and concluded that $\lambda = \sqrt{3/2}$ is the optimal choice.

Growing Trinomial Lattice

A growing trinomial (GT) tree is obtained by setting $m = e^{\nu\triangle t}$. If we define the up and down factors by

$$u = e^{\nu\triangle t + \lambda\sigma\sqrt{\triangle t}}\,, \qquad m = e^{\nu\triangle t}\,, \qquad d = e^{\nu\triangle t - \lambda\sigma\sqrt{\triangle t}}\,, \tag{3.7}$$

then the probabilities are written as

$$p_u = \frac{\hat{\sigma}^2 - (D+1)\,\hat{\sigma}^{1/2} + D}{(U-D)\,(U-1)}\,, \quad p_m = 1 - p_u - p_d\,, \quad p_d = \frac{\hat{\sigma}^2 - (U+1)\,\hat{\sigma}^{1/2} + U}{(U-D)\,(1-D)}\,, \tag{3.8}$$

where U, D are rescaled up and down factors, $U = e^{\lambda\sigma\sqrt{\triangle t}}$, $D = e^{-\lambda\sigma\sqrt{\triangle t}}$. Noticing that the up and down factors for $\lambda = 1$ coincide with those of the JR lattice specification (2.18), this model corresponds to a trinomial-tree extension of the JR binomial model.

Tian

In the paper by Tian [Tia93] a trinomial version of the higher moment matching tree is also proposed. For the trinomial model the third moments match if

$$p_u u^3 + p_m m^2 + p_d d^3 = e^{3(r+\sigma^2)\triangle t}\,. \tag{3.9}$$

Setting the equal probabilities, $p_u = p_m = p_d = 1/3$, then we obtain

$$\begin{gathered} u = a + \sqrt{a^2 - m^2}\,, \quad m = \frac{\hat{r}\,(3-\hat{\sigma})}{2}\,, \quad d = a - \sqrt{a^2 - m^2}\,, \\ p_u = p_m = p_d = \frac{1}{3}\,, \end{gathered} \tag{3.10}$$

where

$$a = \frac{\hat{r}\,(\hat{\sigma}+3)}{4}\,, \tag{3.11}$$

and $\hat{r}$, $\hat{\sigma}$ are defined in the binomial case (2.20).

Log-transformed Trinomial lattice

We introduce another lattice specification based on convergence and stability arguments which were originally made for finite-difference methods of solving the Black-Scholes partial differential equation.

As in the case of the binomial model, an improvement is expected with the log-transformed asset price $x = \log S_t$. The moment matching conditions in the log space are written as

$$p_u \triangle x_u + p_m \triangle x_m + p_d \triangle x_d = \nu \triangle t\,, \qquad p_u \triangle x_u^2 + p_m \triangle x_m^2 + p_d \triangle x_d^2 = \sigma^2 \triangle t + \nu^2 \triangle t^2\,. \tag{3.12}$$

Once the equal jumps $\triangle x \equiv \triangle x_u = -\triangle x_d$ are assumed, we have $\triangle x_m = 0$ from the recombining condition (3.4). Then the probabilities are expressed in terms of $\triangle x$,

$$\begin{aligned} p_u = \frac{1}{2}\left(\frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2} + \frac{\nu \triangle t}{\triangle x}\right)\,, \qquad p_m = 1 - \frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2}\,, \\ p_d = \frac{1}{2}\left(\frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2} - \frac{\nu \triangle t}{\triangle x}\right)\,, \end{aligned} \tag{3.13}$$

Then the current value of the option price is obtained by solving the equation (3.3) recursively,

$$\begin{aligned} C_{i,j} = e^{-r\triangle t}\Bigg[\frac{1}{2}\left(\frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2} + \frac{\nu \triangle t}{\triangle x}\right) C_{i+1,j+1} + \left(1 - \frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2}\right) C_{i+1,j} \\ + \frac{1}{2}\left(\frac{\sigma^2\triangle t + \nu^2 \triangle t^2}{\triangle x^2} - \frac{\nu \triangle t}{\triangle x}\right) C_{i+1,j-1}\Bigg]\,. \end{aligned} \tag{3.14}$$

This is indeed analogous to an explicit finite difference scheme of the Black-Scholes equation discretsing the space and time into the intervals $\triangle t$, $\triangle x$.[5] It should be noted that the space step $\triangle x$ can not be chosen independently of the time step $\triangle t$. The convergence and stability analysis in the explicit finite-difference scheme shows that the spatial and temporal steps should satisfy the following condition,

$$\alpha \equiv \frac{\triangle x^2}{\sigma^2 \triangle t} \geq 3\,. \tag{3.15}$$

Furthermore Ritchken demonstrated in [Rit95] that the optimal choice of α is its minimum value. Hence we shall employ

$$\triangle x = \sigma\sqrt{3\triangle t}\,. \tag{3.16}$$

We denote this lattice specification as LT.

3.2 Dividend

Previously we have handled dividend payments by the shift of the risk-free interest rate for continuous dividend and modification of spot asset price for discrete dividends. As these approaches are independent of the option type and the shape of lattice, one can apply the same techniques to the trinomial tree model. See section 2.2 for detail.

3.3 Time-varying Volatility

For time-dependent volatility and drift we follow the argument on the binomial model. The moment matching conditions corresponding to (2.36) now take the form,

$$p_u^i \triangle x_i - p_d^i \triangle x_i = \nu_i \triangle t_i\,, \qquad p_u^i \triangle x_i^2 + p_d^i \triangle x_i^2 = \sigma_i^2 \triangle t_i + \nu_i^2 \triangle t_i^2\,. \tag{3.17}$$

[5] For finite-difference approaches to the Black-Scholes equation see [Whi13] and references therein, and the relation to the binomial and trinomial models is detailed in [Jam03].

We should note that p_u^i, p_d^i are not constrained in the present case but free parameters to be fixed as far as the condition $0 < p^i < 1$ is satisfied for all of the up, middle and down probabilities. If we rewrite the equations (3.17) as

$$p_u^i = \frac{1}{2}\left(\frac{\sigma_i^2 \triangle t_i + \nu_i^2 \triangle t_i^2}{\triangle x_i^2} + \frac{\nu_i \triangle t_i}{\triangle x_i}\right), \quad p_d^i = \frac{1}{2}\left(\frac{\sigma_i^2 \triangle t_i + \nu_i^2 \triangle t_i^2}{\triangle x_i^2} - \frac{\nu_i \triangle t_i}{\triangle x_i}\right), \tag{3.18}$$

and use the approximation as before $\triangle t_i \to \overline{\triangle t}$, $\nu_i \to \overline{\nu}$, $\sigma_i \to \overline{\sigma}$ for all i, then the error is the same as (2.39) independent of how to approximate $\triangle x_i$. Thus we shall fix $\triangle x_i \to \overline{\triangle x}$ by approximating the third oder moment matching condition,

$$\triangle x_i^3\, p_u^i - \triangle x_i^3\, p_d^i = \nu_i^3 \triangle t_i^3 + 3\sigma_i^2 \nu_i \triangle t_i^2\,. \tag{3.19}$$

The replacement of $\triangle x_i$, $\triangle t_i$, ν_i and σ_i leads to

$$\overline{\triangle x} = \sqrt{3\,\overline{\sigma}^2 \overline{\triangle t} + \overline{\nu}^2 \overline{\triangle t}^2}\,, \tag{3.20}$$

where $\overline{\nu}$ and $\overline{\sigma}$ are defined in (2.38).

Numerical result for this lattice should be compared with the time-varying lattice in the binomial model. As mentioned earlier in section 2.3 significant improvement is not achieved by using the trinomial model. An alternative strategy to the nonconstant parameters is an implied tree discussed in section 4.4.

3.4 Two-dimensional Trinomial Tree Model

For the binomial lattice, the model parameters in the two-dimensional extension are determined uniquely up to the leading order in $1/N$. However, the trinomial lattice possesses more degrees of freedom and free parameters remain after all of the constraints are imposed. Our construction of a two-dimensional model here is such that the one-dimensional KR model is recovered when integrating out one of the two assets,

$$\begin{gathered} u^I = e^{\lambda\sigma^I\sqrt{\triangle t}}\,, \quad m^I = 1\,, \quad d^I = e^{-\lambda\sigma^I\sqrt{\triangle t}}\,, \\ p_{\mu\nu} = \frac{1}{3}p_\mu^1 + \frac{1}{3}p_\nu^2 - \frac{1}{9} + \frac{1}{4}c_{\mu\nu}\rho\,, \end{gathered} \tag{3.21}$$

where μ, ν take u, m, d and p_μ^1, p_ν^2 are one-dimensional trinomial lattice specification (3.6). $c_{\mu\nu}$ are

$$c_{\mu\nu} = \begin{cases} 1 & \mu = \nu \neq m \\ -1 & \mu \neq \nu \text{ and } \mu \neq m \text{ and } \nu \neq m \\ 0 & \mu = m \text{ or } \nu = m \end{cases} \tag{3.22}$$

Although it is true that the trinomial model provides the finer mesh and a better approximation than the binomial model, usage of the two-dimensional trinomial model is limited due to the increased computational effort.

3.5 Hedge Sensitivities

Estimation of Greek values is similar to the binomial model in section 2.5. Since three distinct nodes lie in the layer of $t = \triangle t$, it is possible to evaluate all of the three Greeks without using

the information in the next layer $t = 2\triangle t$,

$$\Delta = \frac{\partial C}{\partial S} \approx \frac{1}{2}\left(\frac{C_{1,2} - C_{1,1}}{S_{1,2} - S_{1,1}} + \frac{C_{1,1} - C_{1,0}}{S_{1,1} - S_{1,0}}\right) . \tag{3.23}$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \approx \frac{\frac{C_{1,2}-C_{1,1}}{S_{1,2}-S_{1,1}} - \frac{C_{1,1}-C_{1,0}}{S_{1,1}-S_{1,0}}}{\frac{1}{2}\left(S_{1,2} - S_{0,0}\right)} . \tag{3.24}$$

$$\Theta = \frac{\partial C}{\partial T} \approx \frac{C_{1,1} - C_{0,0}}{\triangle t} . \tag{3.25}$$

For a growing tree an modification of the formula (2.48) admits an improved estimation for Θ.

In the lattice model various ways of estimation can be considered reflecting the fact that the model involves more adjacent nodes than the binomial tree. For instance an alternative for delta is

$$\Delta = \frac{\partial C}{\partial S} \approx \frac{C_{1,2} - C_{1,0}}{S_{1,2} - S_{1,0}} , \tag{3.26}$$

which in general produces a different value from (3.23). It is not obvious which choice produces a better estimation, rather this depends on what lattice and what sample space we are using. To avoid this ambiguity one can rely on the central finite difference approximation.

3.6 Numerical Results

We present numerical results for the trinomial tree models and discuss the convergence behaviour based on the normalised error defined in (2.54). Our parameter choice is exactly the same as the binomial tree cases for the respective options, but the figures are shown with a wider range of the number of time steps, $(50, 649)$, for the one-dimensional cases.

The European vanilla call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ is shown in Figure 12 for the option price and in Figure 13 for the option delta. Different from the binomial models, the significant odd-even phenomenon are not found but we obtain mild oscillations specific to the lattice specifications. It should be noted that the GR and Tian lattice show exactly the same convergence behaviour, and in fact, this is true for nonstandard options. Although one concludes that the difference in the two lattices, (3.7), (3.8) and (3.10), is a higher order terms in $\triangle t$ and suppressed in option pricing, the Tian lattice is widely used as a starting point of applying advanced techniques (see section 4.3).

The European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and the barrier $H = 25$ is depicted in Figure 14. The jagged curves are obtained for the KR and LT lattices whereas a slow convergence with mild oscillation is found for the other two. The former two lattice specifications have the property of $m = 1$ and this explains the peculiar convergence behaviour. If $S_{i,j}$ is equal to or slightly less than the barrier H, the effective barrier in the lattice is located at $S_{i+k,j+k}$ at every time step and is close enough to H. Conversely, if $S_{i,j}$ is slightly larger than H, the effective barrier in the lattice is at $S_{i+k,j+k-1}$ away from the true barrier H, then the nonlinearity error associated with the knock-out feature becomes significantly large. Proposed solution to this problem is to adjust λ such that the effective barrier is located at the true barrier H, which is discussed in detail in section 4.2.

For the American vanilla put option with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ we illustrate the normalised error in Figure 15. Besides the absence of the odd-even property, the fast convergence is obtained except the LT lattice for this parameter set. One lessen from the

comparison between the European option in Figure 12 and the American option in Figure 15 is that the respective lattice specifications exhibit very different convergence behaviour for these two options. Thus the fast convergence for European options is not a strong indication that using the same lattice is optimal for American options.

We apply the two-dimensional trinomial model to the exchange-one-asset-for-another option defined in (2.56) with exactly the same set of the parameters as before, $S_0^1 = 29$, $S_0^2 = 31$, $T = 1$, $\sigma_0^1 = 0.3$, $\sigma_0^2 = 0.2$, $r = 0.1$, $\rho = 0.6$, $Q^1 = Q^2 = 1$. Reflecting the increased number of nodes, the slight improvement is achieved in the trinomial model in Figure 16.

4 Advanced Techniques in Tree Models

As shown in sections 2.6 and 3.6 the tree models suffer from the oscillatory convergence that prevents one to estimate precision of computation. This is due to the nonlinearity error concerning relative positions between the nodes and the strike, the barrier and so on. Several techniques are proposed in order to remove the nonlinearity error and achieve the smooth convergence.

In section 4.1 we discuss flexible binomial tree in which the degree of freedom λ is chosen such that one node at expiry exactly matches the strike price.

The nonlinearity error arising from the barrier can be handled in the trinomial model by using adaptive mesh method. In section 4.2 we consider simple approaches proposed in [Rit95] and [FG99]. Similar to the flexible binomial tree the extra degree of freedom is used such that the nodes are on the barrier.

Another approach to smooth convergence is introduced in [BD96] as a part of acceleration technique for American option valuation. Section 4.3 discusses the acceleration together with other useful techniques for an American put option including Richardson extrapolation and truncation of the binomial tree.

Nonconstant feature of the model parameters discussed in sections 2.3 and 3.3 is more nicely captured within the framework of the recombinant tree by building a trinomial tree consistent with market prices of European options. This is detailed in section 4.4

4.1 Flexible Binomial Tree

The flexible binomial tree was proposed in [Tia99] and the proof of its smooth convergence was given for European vanilla option and digital option in [CP07]. Set $u_0 = e^{\sigma\sqrt{\triangle t}}$, $d_0 = u_0^{-1}$ and find j_0 at expiry such that

$$S_0 u_0^{j_0-1} d_0^{N-j_0+1} < K < S_0 u_0^{j_0} d_0^{N-j_0} \,, \tag{4.1}$$

then the up factor and down factor are defined as

$$u = \exp\left(\sigma\sqrt{\triangle t} + \lambda\sigma^2\triangle t\right) \,, \qquad d = \exp\left(\sigma\sqrt{\triangle t} - \lambda\sigma^2\triangle t\right) \,, \tag{4.2}$$

where λ solves

$$K = S_0 u_0^{j_0} d^{N-j_0} \,. \tag{4.3}$$

While rigorous argument of the smooth convergence is given only for limited options, it is indeed useful for other types including American options since the adjustment of the lattice resolves at least one source of the nonlinearity errors.

The convergence behaviour of the flexible lattice is illustrated for the European vanilla call option in Figure 17 and for the American vanilla put option in Figure 18. Here we use the same parameter sets as before, i.e., $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the European option and $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the American option. These results should be compared with the binomial models in Figures 1-3 and Figures 9, 10.

4.2 Adaptive Mesh Methods

For vanilla options the flexible binomial tree model and acceleration technique are clues for achieving smooth convergence (sections 4.1 and 4.3). For barrier options, however, the barrier is another source of the nonlinearity error and the convergence still exhibits the sawtooth pattern even after the nonlinearity error at expiry is handled.

In this section we review the adaptive mesh method to the barrier options proposed in [Rit95] and improved in [FG99]. The former work demonstrated that computation of the single barrier option price in the trinomial model is greatly improved by adjusting the lattice such that a layer of nodes coincides with the barrier. Under the assumption that an option has a single barrier H which is constant in time, we start with the specification (3.13) with $\triangle x = \lambda\sigma\sqrt{\triangle t}$ and find λ, j such that $||3 - \lambda||$ takes the minimum value and

$$S_0 e^{j\lambda\sigma\sqrt{\triangle t}} = H\,, \tag{4.4}$$

for $-N < j < N$. If j is not within this range, the barrier lies outside of the trinomial tree. With this adjustment of λ the barrier coincides with one node in each layer after $t = j\triangle t$.

In this approach we encounter the difficulty in the case the current asset price is close to the barrier. If the spot and barrier are so close that they are apart only by one space step, for instance, $S_0 e^{-\triangle x} \leq H < S_0$ for the down-and-out case, the probability to hit the barrier is too high, and consequently, the option price tends to be undervalued. To insert extra nodes between H and S_0 by increasing N sometimes requires the huge number of the time steps. In [FG99] this issue is addressed introducing fine resolution of the lattice only in the vicinity of the barrier, that allows one to improve the computation result without changing N. The model parameters for the fine mesh are obtained from (3.13) by adjusting $\triangle x$ and $\triangle t$. Then the local mesh is consistently connected to the original lattice nodes.

The adaptive mesh methods above are not applicable to double barrier options since no extra degree of freedom remains for synthesising location of node points to the other barrier. A new approach to the valuation of these securities is suggested in [DL10] by using a combination of binomial and trinomial trees.

The normalised error defined in (2.54) for the down-and-out European vanilla call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$, $H = 25$ is shown as a function of number of time steps in Figure 19. Because the spot is sufficiently distinct from the barrier, we only apply the version of [Rit95] to the trinomial lattice. Compared with the standard approaches in Figures 7, 8 and 14 it is clear that the convergence behaviour is remarkably improved with the adaptive mesh as a consequence of the fact that the nonlinearity error of the barrier is resolved. Remaining approximation errors in the payoff function at expiry cause small jumps in the graph, but we see that the errors at expiry are much smaller than those associated with the barrier.

4.3 Acceleration and Truncation

Since early exercise is never optimal for an American call option without dividend payment, we consider an American put option on an asset without dividend payment as the simplest example

of the early exercise feature.

According to [Sta05] tree models for the American put option pricing are more efficient than finite difference methods in the PDE approach, and a recent paper [CJTY09] demonstrates that the binomial model has advantages to the trinomial model. In [CJ12] various speed-up techniques are compared for the binomial model. Our objective in this section is to discuss acceleration and truncation techniques based on the Tian binomial lattice.

The acceleration consists of two procedures, smoothing and Richardson extrapolation. In [BD96] the acceleration technique was first applied to the binomial model. There are several ways to achieve the smooth convergence and one simple method which works effectively for the American option is to replace the option values in the second last layer by the Black-Scholes prices. By taking N to be large enough one can avoid the option has exercise opportunity after the second last step, then this replacement is a good approximation. For a smoothly converging sequence Richardson extrapolation can be applied. The expansion formula (2.26) is now expressed with a constant F,

$$P(N) = \text{Exact} + \frac{F}{N} + o\left(\frac{1}{N}\right) , \tag{4.5}$$

The independence of N in F allows one to cancel out the leading-order error term by computing $P(N/2)$ for even N,

$$2P(N) - P(N/2) = \text{Exact} + o\left(\frac{1}{N}\right) . \tag{4.6}$$

By a similar manipulation the leading error terms cancel out for odd N.

In the i-th layer the probability for the asset price to take a value $S_{i,j}$ is given by

$$p^{i,j} = \begin{pmatrix} i \\ j \end{pmatrix} p_u^j \, p_d^{i-j} . \tag{4.7}$$

Recalling that p_u, p_d are smaller than 1 and the value of the put option at (i, j), $P_{i,j}$, is bounded from above by the strike price, the contribution to the total option price, $P_{i,j}\, p^{i,j}$, tends to become very small for large i with large j or small j. In fact only nodes with $S_{\min} \leq 0 \leq S_{\max}$ are relevant to the option valuation and various strategies to determine the bounds $S_{\min}$, $S_{\max}$ are compared in [AWDN04]. One of the truncation methods which offer stable option valuation for both of the positive and negative drifts is driven by

$$S_{\max} = Ke^{-r(T-i\triangle t)+\xi\sigma\sqrt{T-i\triangle t}} , \qquad S_{\min} = Ke^{-r(T-i\triangle t)-\xi\sigma\sqrt{T-i\triangle t}} , \tag{4.8}$$

and option values at the truncated notes are replaced by intrinsic values at the respective nodes. Other ways of the truncation are studied in [CJ12]. In (4.8) we introduced the parameter ξ which controls the truncation positions. Fixing the truncation introduces another lower bounded for the approximation error, and then, improvement of the computational results becomes limited even if a large N is used. Thus the choice of ξ should be synthesised with objective numerical accuracy.

Figure 20 shows the normalised error (2.54) as a function of computation time (nano second) for the American put option with the same parameter set $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$. Here we compare the result using the acceleration and truncation with that of the Tian trinomial model (3.10), and find that 10-100 times accuracy is realised for a given computational time. Note that the truncation method is not greatly advantageous for small N because of the small number of truncated nodes.

4.4 Implied Trees

In sections 2.3 and 3.3, aiming at the consistency with the observed structure of market implied volatilities, nonconstant volatility was discussed for the binomial and trinomial models where we "smooth out" the time-varying paremeters to keep the recomninat property of the lattices, and it turned out that both the models suffer from the errors which do not decay fast enough in the large N limit. In addition it is a shortcoming that the strike value dependence of the market volatilities is not captured by those models.

Implied tree theories were proposed such that a tree model is consistent with shape of implied volatility smile [DK94, DEK95], and advantages in using the trinomial model were proposed in [Dup94]. Their strategy is to construct a inhomogeneous trinomial tree, i.e., tree associated with node-specific transition probabilities, $p_u^{i,j}$, $p_m^{i,j}$, $p_d^{i,j}$, such that European vanilla option value at each node $C(S_{i,j-1}, i\triangle t)$ is consistent with the market value. We shall briefly review how to determine the probabilities $p_u^{i,j}$, $p_m^{i,j}$, $p_d^{i,j}$ from the option market data.

Let $Q_{i,j}$ to be Arrow-Debreu (AD) security at node (i,j), i.e., price today of an instrument whose payoff at $t = i\triangle t$ is unity if the node (i,j) is reached. Then the call option price with strike K and expiry $i\triangle t$ is written as

$$C(K, i\triangle t) = \sum_{j=0}^{2i} \max\left(S_{i,j} - K\right) Q_{i,j}\,. \tag{4.9}$$

Especially if we choose the strike price as a set of possible terminal asset prices $S_{i,j}$, the contributions blow the j-th level are zero. Then we have

$$C(S_{i,j-1}, i\triangle t) = \sum_{k=j}^{2i} \left(S_{i,k} - S_{i,j-1}\right) Q_{i,k}\,. \tag{4.10}$$

Conversely, if the option prices $C(S_{i,j-1}, i\triangle t)$ are known from the market data, the price of the AD securities with the same expiry, $Q_{i,j}$, is derived by solving (4.10) iteratively,

$$Q_{i,j} = \frac{C(S_{i,j-1}, i\triangle t) - \displaystyle\sum_{k=j+1}^{2i} \left(S_{i,k} - S_{i,j-1}\right) Q_{i,k}}{S_{i,j} - S_{i,j-1}}\,. \tag{4.11}$$

Starting with the highest node $(i, 2i)$ we compute $Q_{i,j}$ at every node up to $j = i$, and determine the remaining AD security values from the lowest node $(i, 0)$ by using the put option prices in order to avoid accumulated numerical errors.

The next step is to relate the AD securities to the transition probabilities. To accomplish this we shall impose the probability conservation, risk-neutrality condition for the underlying and risk-neutrality condition for the AD security,

$$\begin{aligned}
&p_u^{i,j} + p_m^{i,j} + p_d^{i,j} = 1\,,\\
&p_u^{i,j} S_{i+1,j+2} + p_m^{i,j} S_{i+1,j} + p_d^{i,j} S_{i,j} = S_{i,j} e^{r\triangle t}\,,\\
&p_u^{i,j-1} Q_{i,j-1} + p_m^{i,j} Q_{i,j} + p_d^{i,j+1} Q_{i,j+1} = Q_{i+1,j+1} e^{r\triangle t}\,.
\end{aligned} \tag{4.12}$$

The system has the unique solution, and the up, middle and down probabilities are written as

$$
\begin{aligned}
p_u^{i,j} &= \frac{e^{r\triangle t}Q_{i+1,j+2} - p_d^{i,j+2}Q_{i,j+2} - p_m^{i,j+1}Q_{i,j+1}}{Q_{i,j}}\,, \\
p_m^{i,j} &= \frac{e^{r\triangle t}S_{i,j} - S_{i+1,j} - p_u^{i,j}\left(S_{i+1,j+2} - S_{i+1,j}\right)}{S_{i+1,j+1} - S_{i+1,j}}\,, \\
p_d^{i,j} &= 1 - p_u^{i,j} - p_m^{i,j}\,.
\end{aligned}
\tag{4.13}
$$

The up transition probabilities for the highest node and the second highest node are respectively obtained by

$$
p_u^{i,2i} = \frac{Q_{i+1,2i+2}e^{r\triangle t}}{Q_{i,2i}}\,, \qquad p_u^{i,2i-1} = \frac{Q_{i+1,2i+1}e^{r\triangle t} - p_m^{i,2i}Q_{i,2i}}{Q_{i,2i-1}}\,. \tag{4.14}
$$

Hence, once the upper probabilities are determined, one can use the formulas in (4.13) for their middle and down probabilities.

Generally the derived transition probabilities are not positive-definite. One origin of the negative probability is a large value of the local volatility at a tree node. In fact, with a large call option price $C(S_{i,j-1}, i\triangle t)$ we have a large $Q_{i,j}$, and accordingly, a large up probability $p_u^{i,j}$. This tends to produce negative middle and down probabilities.[6] If the possibilities are negative, they are modified as suggested in [DKC96],

$$
\begin{aligned}
p_u^{i,j} &= \frac{1}{2}\left(\frac{S_{i,j}e^{r\triangle t} - S_{i+1,j+1}}{S_{i+1,j+2} - S_{i+1,j+1}} + \frac{S_{i,j}e^{r\triangle t} - S_{i+1,j}}{S_{i+1,j+2} - S_{i+1,j}}\right)\,, \\
p_d^{i,j} &= \frac{1}{2}\frac{S_{i+1,j+2} - S_{i,j}e^{r\triangle t}}{S_{i+1,j+2} - S_{i+1,j}}\,,
\end{aligned}
\tag{4.15}
$$

for $S_{i+1,j+1} < S_{i,j}e^{r\triangle t} < S_{i+1,j+2}$ or

$$
\begin{aligned}
p_u^{i,j} &= \frac{1}{2}\frac{S_{i,j}e^{r\triangle t} - S_{i+1,j}}{S_{i+1,j+2} - S_{i+1,j}}\,, \\
p_d^{i,j} &= \frac{1}{2}\left(\frac{S_{i+1,j+2} - S_{i,j}e^{r\triangle t}}{S_{i+1,j+2} - S_{i+1,j}} + \frac{S_{i+1,j+1} - S_{i,j}e^{r\triangle t}}{S_{i+1,j+1} - S_{i+1,j}}\right)\,,
\end{aligned}
\tag{4.16}
$$

for $S_{i+1,j} < S_{i,j}e^{r\triangle t} < S_{i+1,j+1}$. The middle probability is determined from the probability conservation.

Another market-implied tree is constructed within the framework of the binomial model [Rub94], where the tree is synthesised to market values of European call and put options whose expiry is the same as that of the target security. Thus the implied binomial tree is consistent with the market only at expiry in contrast with that the market consistency holds in the implied trinomial tree in every layer of the time steps.

[6] In addition the negativity of the probability is encountered when the forward price of the underlying $e^{r\triangle t}S_{i,j}$ is outside the sample space spanned by the asset prices in the next layer, $S_{i+1,j}$, $S_{i+1,j+1}$ and $S_{i+1,j+2}$. One can avoid this situation with an appropriate choice for the sample space [DKC96].

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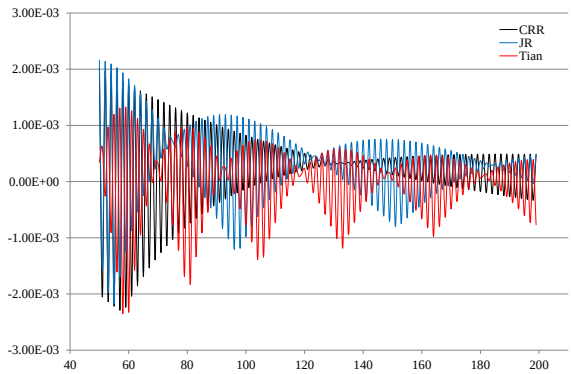

Figure 1: Convergence to the analytic price formula of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the CRR binomial lattice (black line), the JR binomial lattice (blue line) and the Tian binomial lattice (red line).

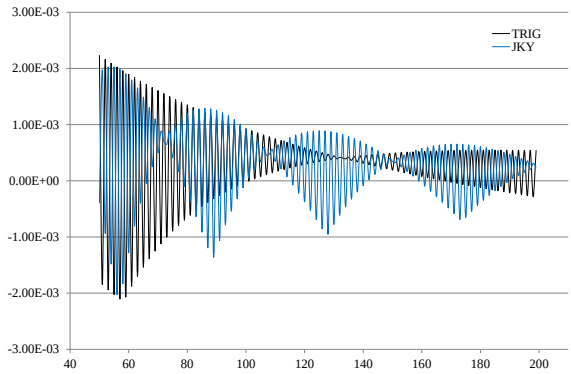

Figure 2: Convergence to the analytic price formula of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the Trigeorgis binomial lattice (black line) and the JKY binomial lattice (blue line).

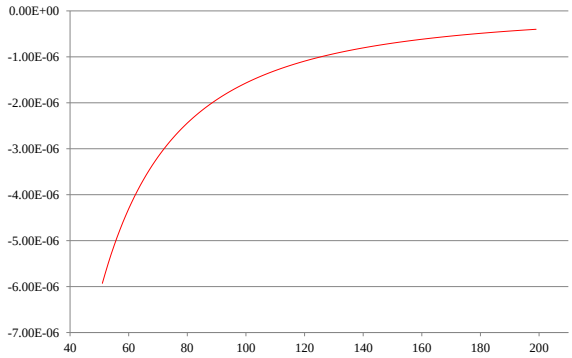

Figure 3: Convergence to the analytic price formula of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the LR binomial lattice.

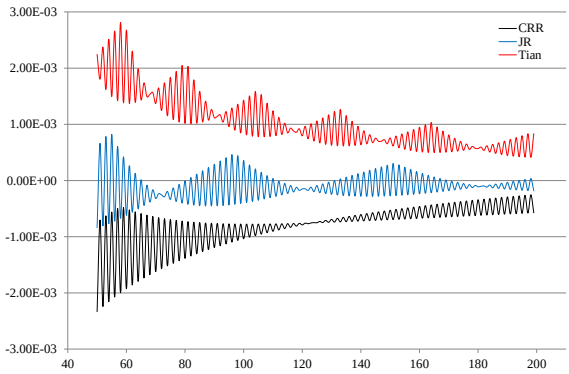

Figure 4: Convergence to the Black-Scholes delta of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the CRR binomial lattice (black line), the JR binomial lattice (blue line) and the Tian binomial lattice (red line).

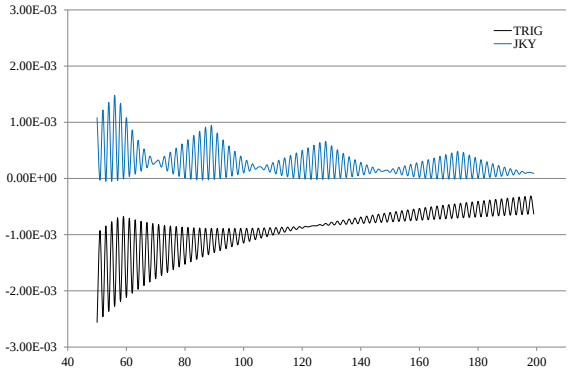

Figure 5: Convergence to the Black-Scholes delta of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the Trigeorgis binomial lattice (black line) and the JKY binomial lattice (blue line).

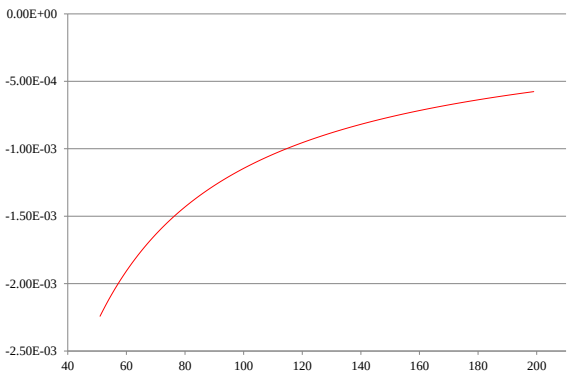

Figure 6: Convergence to the Black-Scholes delta of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the LR binomial lattice.

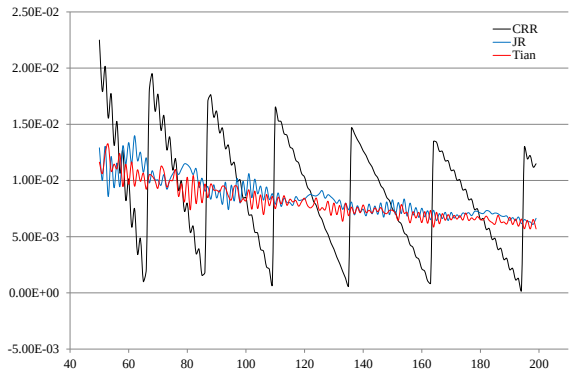

Figure 7: Convergence to the analytic price formula of the European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and $H = 25$ for the CRR binomial lattice (black line), the JR binomial lattice (blue line) and the Tian binomial lattice (red line).

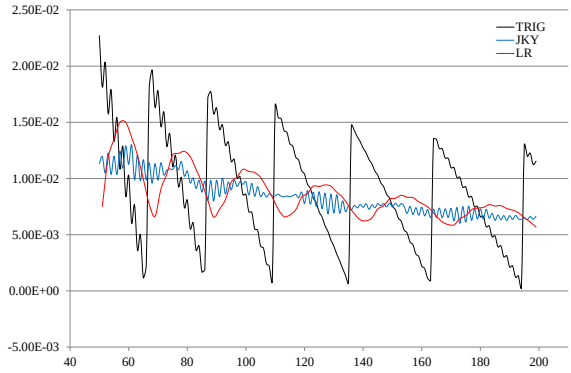

Figure 8: Convergence to the analytic price formula of the European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and $H = 25$ for the Trigeorgis binomial lattice (black line), the JKY binomial lattice (blue line) and the LR binomial lattice (red line).

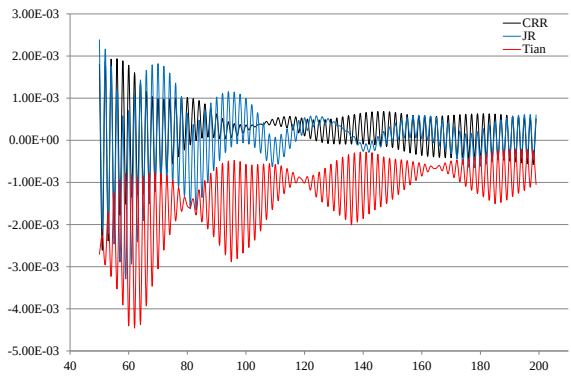

Figure 9: Convergence of the American vanilla put option price with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the CRR binomial lattice (black line), the JR binomial lattice (blue line) and the Tian binomial lattice (red line).

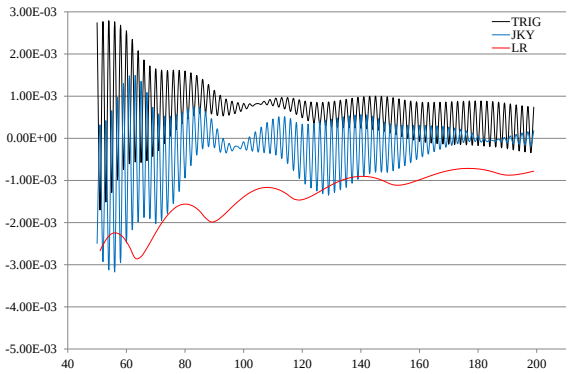

Figure 10: Convergence of the American vanilla put option price with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the Trigeorgis binomial lattice (black line), the JKY binomial lattice (blue line) and the LR binomial lattice (red line).

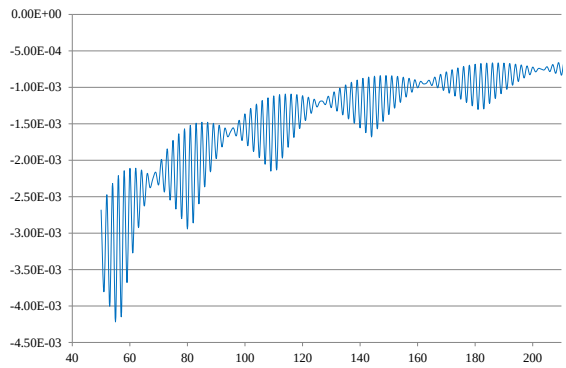

Figure 11: Convergence to the analytic price formula of the exchange-one-asset-for-another option with $S_0^1 = 29$, $S_0^2 = 31$, $T = 1$, $\sigma_0^1 = 0.3$, $\sigma_0^2 = 0.2$, $r = 0.1$, $\rho = 0.6$, $Q^1 = Q^2 = 1$ for the two-dimensional binomial model.

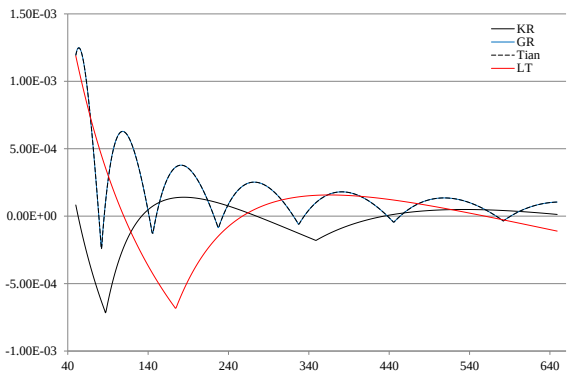

Figure 12: Convergence to the analytic price formula of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the KR lattice (black line), the growing trinomial lattice (blue line), Tian trinomial lattice (black dashed line) and the log-transformed trinomial lattice (red line).

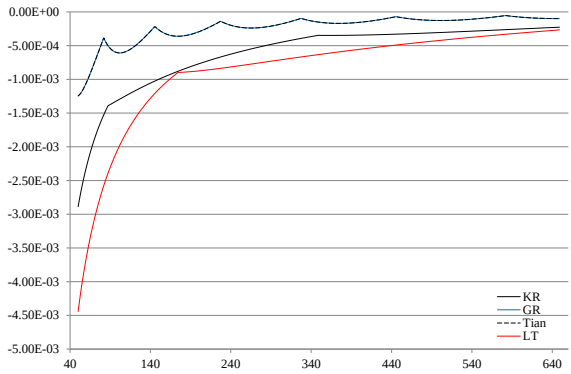

Figure 13: Convergence to the Black-Scholes delta of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the KR lattice (black line), the growing trinomial lattice (blue line), Tian trinomial lattice (black dashed line) and the log-transformed trinomial lattice (red line).

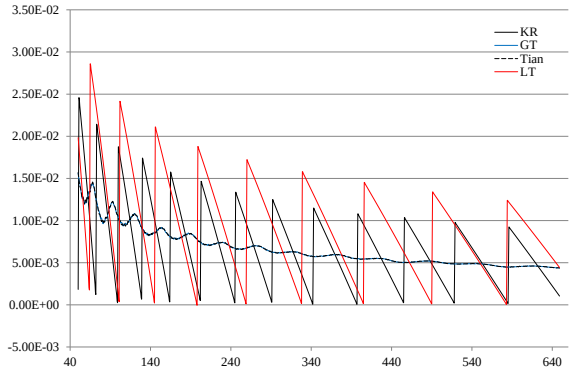

Figure 14: Convergence to the analytic price formula of the European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and $H = 25$ for the KR lattice (black line), the growing trinomial lattice (blue line), Tian trinomial lattice (black dashed line) and the log-transformed trinomial lattice (red line).

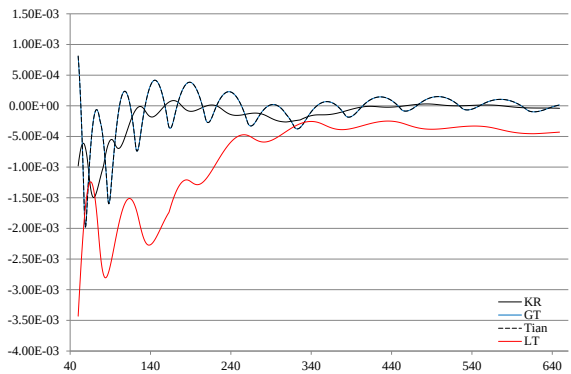

Figure 15: Convergence of the American vanilla put option price with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for KR lattice (black line), the growing trinomial lattice (blue line), Tian trinomial lattice (black dashed line) and the log-transformed trinomial lattice (red line).

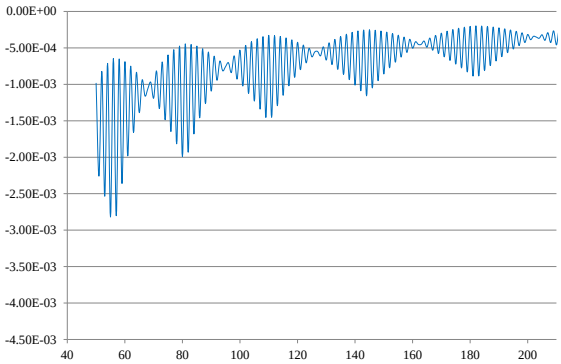

Figure 16: Convergence to the analytic price formula of the exchange-one-asset-for-another option with $S_0^1 = 29$, $S_0^2 = 31$, $T = 1$, $\sigma_0^1 = 0.3$, $\sigma_0^2 = 0.2$, $r = 0.1$, $\rho = 0.6$, $Q^1 = Q^2 = 1$ for the two-dimensional trinomial model.

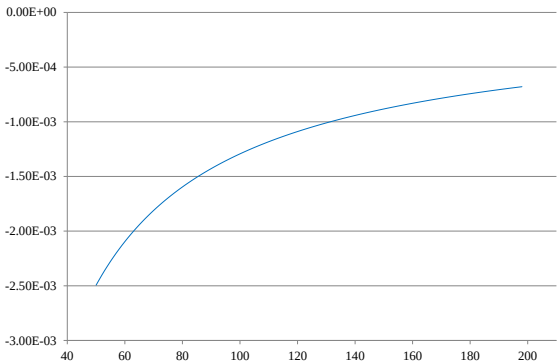

Figure 17: Convergence to the analytic price formula of the European vanilla call option with $S = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the flexible binomial lattice.

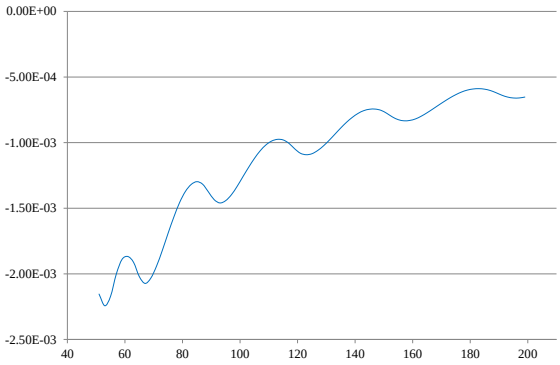

Figure 18: Convergence of the American vanilla put option price with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ for the flexible binomial lattice.

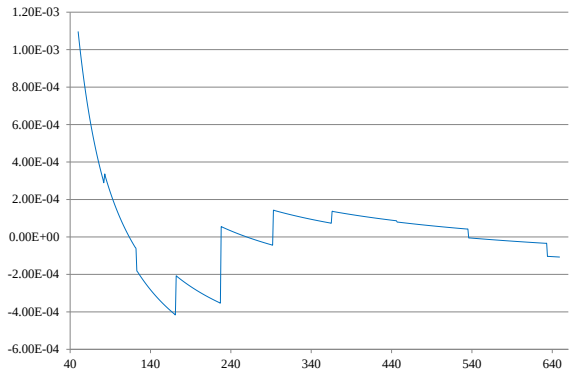

Figure 19: Convergence to the analytic price formula of the European-type down-and-out call option with $S_0 = 31$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$ and $H = 25$ for the adaptive mesh.

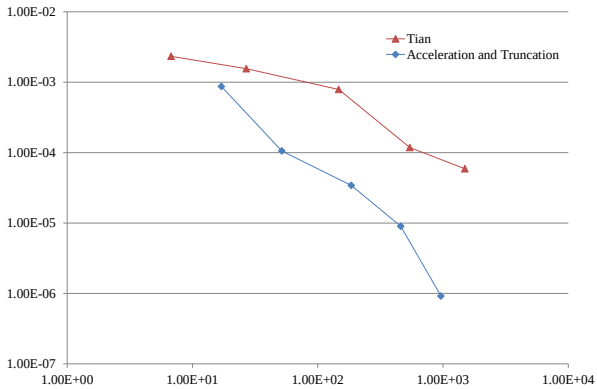

Figure 20: The normalised error as a function of computation time (nano second) for the American put option with $S_0 = 29$, $K = 30$, $T = 1$, $\sigma = 0.25$, $r = 0.1$. The Tian trinomial lattice is used for the red line whereas the truncation and acceleration techniques are applied to the lattice for the blue line.

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OpenGamma helps financial services firms unify their calculation of analytics across the traditional trading and risk management boundaries.

The company's flagship product, the OpenGamma Platform, is a transparent system for front-office and risk calculations for financial services firms. It combines data management, a declarative calculation engine, and analytics in one comprehensive solution. OpenGamma also develops a modern, independently-written quantitative finance library that can be used either as part of the Platform, or separately in its own right.

Released under the open source Apache License 2.0, the OpenGamma Platform covers a range of asset classes and provides a comprehensive set of analytic measures and numerical techniques.

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