

SMILE EXTRAPOLATION

OPENGAMMA QUANTITATIVE RESEARCH



ABSTRACT. An implementation of smile extrapolation for high strikes is described. The main smile is described by an implied volatility function, e.g. SABR. The extrapolation described is available for cap/floor and swaption pricing in the OpenGamma library.

1. INTRODUCTION

The *smile* is the description of the strike dependency of option prices through Black implied volatilities. The name come from the shape of the curve, which for most markets resembles a smile.

For interest rate markets like cap/floor and swaption, the smile is often described using a model like SABR; an approximated implied volatility function is used to obtain the prices. The model parameters are calibrated to fit the smile for quoted options, which usually have strikes in the at-the-money region.

The standard approximated methods do not produce very good results far away from the money. Moreover, the model calibration close to the money does not necessarily provide relevant information for far away strikes. When only vanilla options are priced, these problems are relatively minor, as the volatility impact on those far away from the money options is small.

This is not the case for instruments where pricing depends on the full smile. In the cap/floor world, these instruments include in-arrears swaps and cap/floors and swaps with long or short tenors. In the swaption world, they include the CMS swap and cap/floor (pre- and post-fixed).

The requirements for the method, in addition to ease of implementation, is it they should provide arbitrage-free extrapolation and leave a degree of freedom on the tail to calibrate to the traded smile-dependent products. The degree of freedom will be referred to as the *tail control* or *tail thickness*.

The implementation described here is based on ?. Call prices are extrapolated; put prices are obtained by put/call parity.

2. NOTATION

The analysis framework is the pricing of options through a Black formula and an implied volatility description. In the central region, the call prices are obtained by the formula

$$(1) \quad C(F, K) = N \cdot \text{Black}(F, K, \sigma(F, K)).$$

where F is the forward, K the strike, N the numeraire and σ the forward- and strike-dependent implied Black volatility.

3. EXTRAPOLATION

A cut-off strike K^* is selected. Below that strike the pricing formula described above is used. Above that strike an extrapolation of call prices on strikes is used. The shape of the extrapolation

is based on prices (and not on volatilities). The functional form of the extrapolation is taken from ? and is given by

$$(2) \quad f(K) = K^{-\mu} \exp \left(a + \frac{b}{K} + \frac{c}{K^2} \right).$$

The parameter μ will be used to control the tail. The other three parameters are used to ensure C^2 regularity of the price with respect to the strike.

3.1. Gluing. The *gluing* between the two parts requires some regularity condition. It is required¹ to be C^2 .

To be able to achieve this smooth gluing, we need to compute the derivatives of the price with respect to the strike up to the second order. The derivatives of the price are, for the first order,

$$D_K C(F, K) = N (D_K \text{Black}(F, K, \sigma(F, K)) + D_\sigma \text{Black}(F, K, \sigma(F, K)) D_K \sigma(F, K))$$

and, for the second order,

$$\begin{aligned} D_{K,K}^2 C(F, K) = & N \left(D_{K,K}^2 \text{Black}(F, K, \sigma) + D_{\sigma,K}^2 \text{Black}(F, K, \sigma) D_K \sigma(F, K) \right. \\ & + (D_{K,\sigma}^2 \text{Black}(F, K, \sigma) + D_{\sigma,\sigma}^2 \text{Black}(F, K, \sigma) D_K \sigma(F, K)) D_K \sigma(F, K) \\ & \left. + D_\sigma \text{Black}(F, K, \sigma(F, K)) D_{K,K}^2 \sigma(F, K) \right). \end{aligned}$$

3.2. Extrapolation derivatives. The functional form of the extrapolation is

$$f(K) = K^{-\mu} \exp \left(a + \frac{b}{K} + \frac{c}{K^2} \right).$$

Its first and second order derivatives are required to ensure the smooth gluing. The derivatives are

$$f'(K) = K^{-\mu} \exp \left(a + \frac{b}{K} + \frac{c}{K^2} \right) (-\mu K^{-1} - b K^{-2} - 2c K^{-3}) = f(K) (-\mu K^{-1} - b K^{-2} - 2c K^{-3})$$

and

$$f''(K) = f(K) \left(\mu(\mu+1) K^{-2} + 2b(\mu+1) K^{-3} + 2c(2\mu+3) K^{-4} + b^2 K^{-4} + 4bc K^{-5} + 4c^2 K^{-6} \right).$$

3.3. Fitting. The two parts are fitted at the cut-off strike. At that strike, the price and its derivatives are $C(F, K^*) = p$, $\partial C / \partial K(F, K^*) = p'$ and $\partial^2 C / \partial K^2(F, K^*) = p''$.

To obtain the extrapolation parameters (a , b , c), a system of three equations with three unknowns has to be solved. Due to the structure of the function, it is possible to isolate the first two variables (a and b) one-by-one and to solve a simpler problem of one equation with one unknown.

The parameter a can be written explicitly in term of b and c :

$$a(b, c) = \ln(p K^\mu) - \frac{b}{K} - \frac{c}{K^2}.$$

The parameter b can be written as function of c :

$$b(c) = -2c K^{-1} - \left(\frac{p'}{p} K + \mu \right) K.$$

The equation to solve is

$$\frac{p''}{p} K^2 - \mu(\mu+1) = 2b(c)(\mu+1) K^{-1} + 2c(2\mu+3) K^{-2} + b(c)^2 K^{-2} + 4b(c)c K^{-3} + 4c^2 K^{-4}.$$

¹Actually, a weak second order condition would be sufficient but could produce a jump in the cumulative density and an atomic weight on the density.

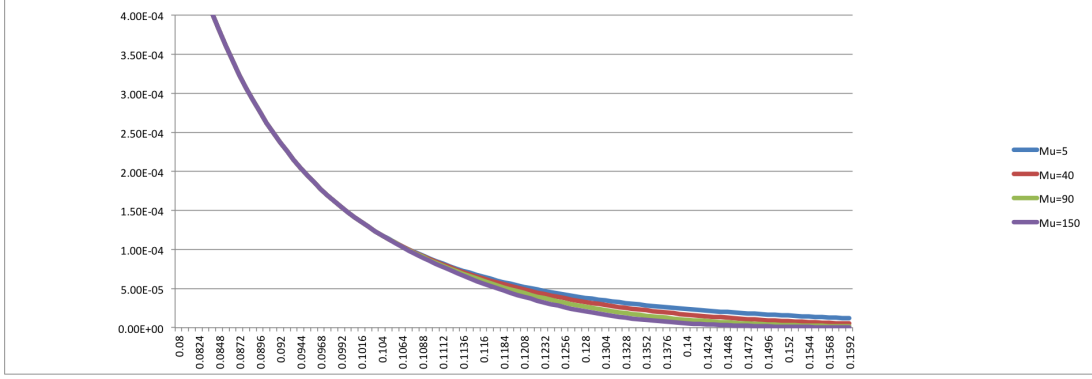


FIGURE 1. Price extrapolation

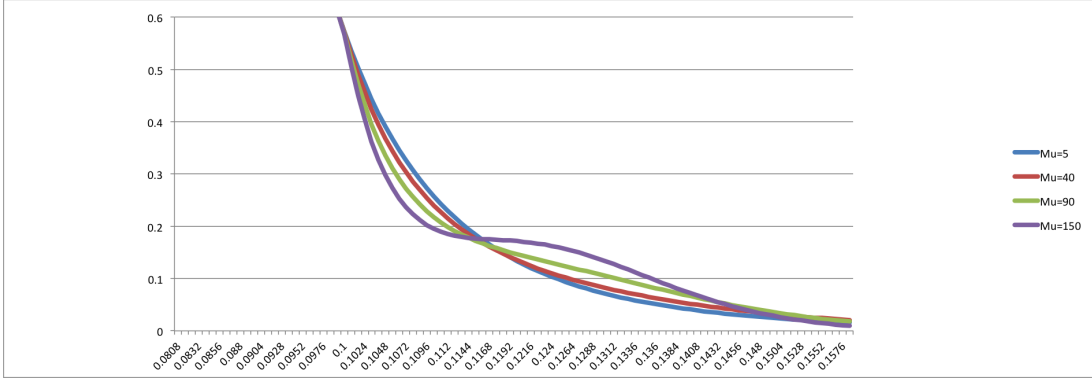


FIGURE 2. Price extrapolation: impact on density

4. EXAMPLES

Some examples of smile extrapolation are presented. The SABR data used is $\alpha = 0.05$, $\beta = 0.50$, $\rho = -0.25$ and $\nu = 0.50$. The forward is 5%, the cut-off strike 10% and the time to expiry 2.

The extrapolation is computed for $\mu = 5, 40, 90$ and 150. The choices of the exponents have been exaggerated to obtain clearer pictures.

In Figure 1 the tail of the price (not adjusted by the numeraire) is given.

In Figure 2 the tail of the price density is given.

In Figure 3 the tail of the smile is given.

The impact of the smile extrapolation on the CMS prices is important. In Table 1, a pre-fixed CMS coupon with approximately 9Y on a 5Y swap is computed. Depending on the tail, the CMS convexity adjustment can be very different (up to 50bps difference). The values of μ in the table have been selected to have differences of around 10 bps in adjusted rate between them.

5. DERIVATIVES

To compute the price derivatives (greeks) with respect to the different market inputs, it is useful to have the derivatives of the building blocks. In this case, we would like to have the derivative of the price $f(K)$ with respect to the forward F and the parameters describing the volatility surface σ (called p hereafter).

From the way the price is written, it may appear that $f(K)$ does not depend on F and p . Actually, the dependency is absorbed into the parameters a , b and c . To see this, we rewrite the equation.

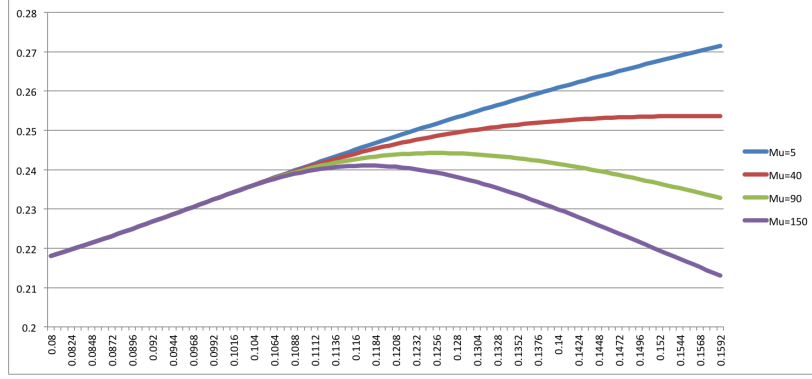


FIGURE 3. Price extrapolation: impact on volatility smile

Method	Parameter (μ)	Adjusted rate (in %)
SABR	—	5.08
SABR with extrapolation	1.30	5.03
SABR with extrapolation	1.55	4.98
SABR with extrapolation	2.25	4.88
SABR with extrapolation	3.50	4.78
SABR with extrapolation	6.00	4.68
SABR with extrapolation	15.00	4.58
No convexity adjustment	—	4.04

TABLE 1. CMS prices with different methods.

Let \tilde{f} denote the vector function (f, f', f'') and \tilde{P} the vector function $(C, \partial C / \partial K, \partial^2 C / \partial K^2)$. The equation solved to obtain $x = (a, b, c)$ is

$$\tilde{f}(K^*, x) = \tilde{C}(F, K^*, p).$$

In what follows, the cut-off strike K^* is a constant and we ignore it to simplify workings. The equation to solve is then

$$g(x, F, p) = \tilde{f}(x) - \tilde{C}(F, p) = 0.$$

The equation is usually solved by numerical techniques. We have only one solution point, but we would like to compute the derivative of the solution around the initial point. We have, at the initial point (F_0, p_0) ,

$$x_0 = x(F_0, p_0)$$

which solves

$$g(x_0, F_0, p_0) = 0$$

and we look for

$$D_F x(F_0, p_0) \quad \text{and} \quad D_p x(F_0, p_0).$$

Unfortunately x is not known explicitly; we only know that such a function $x(F, p)$ should exist around (F_0, p_0) . To get the required derivatives, we rely on the *implicit function theorem* which states the existence of such a function and gives its derivative from g derivatives. If $D_x g(x_0, F_0, p_0)$ is invertible, then the implicit function $x(F, p)$ that solves the equation

$$g(x(F, p), F, p) = 0$$

exists around (F_0, p_0) ,

$$D_F x(F_0, p_0) = -(D_x g(x_0, F_0, p_0))^{-1} D_F g(x_0, F_0, p_0)$$

and

$$D_p x(F_0, p_0) = -(D_x g(x_0, F_0, p_0))^{-1} D_p g(x_0, F_0, p_0).$$

6. IMPLEMENTATION

In the OpenGamma library the extrapolation is implemented in the class `SABRExtrapolationRightFunction`. The call prices are extrapolated and the put prices are obtained by put/call parity.

The extrapolation is used for the swaption pricing in `SwaptionPhysicalFixedIborSABRExtrapolationRightMethod` and the CMS pricing in the class `CouponCMSSABRExtrapolationRightReplicationMethod`.

To implement that formula, the following derivatives should be available:

$$D_K \text{Black}, D_\sigma \text{Black}, D_{K,K}^2 \text{Black}, D_{\sigma,K}^2 \text{Black}, D_{\sigma,\sigma}^2 \text{Black}$$

$$D_K \sigma, D_{K,K}^2 \sigma$$

The required partial derivatives of the Black formula and of the ? SABR functional formula are also available in the library. The Black formula is available in the class `BlackPriceFunction` and the partial derivatives of first and second order are available in the method `getPriceAdjoint2`. The SABR approximated formula is available in the class `SABRHaganVolatilityFunction` and the required first and second order derivatives are in the method `getVolatilityAdjoint2`.

For the Black formula, the computation time for the three first order derivatives (forward, strike, volatility) and three second order derivatives (strike-strike, strike-volatility, volatility-volatility) is around 1.75 times the price time. A finite difference implementation would require around 7 times the time of a price and provide less stability.

Obviously, the implementation with extrapolation will be slower than the implementation without extrapolation. For swaptions, the SABR with extrapolation takes around twice the time for the standard SABR in the extrapolated region.

REFERENCES

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