MY FUTURE IS NOT CONVEX

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Abstract. We propose a framework where interest rate futures pricing do not require convexity adjustment. The adjustment depends on the definition of curves and we build them in such a way that no adjustment is necessary. The framework is theoretically as acceptable as the standard (current) approach and may prove in some circumstances simpler to work with in practice.

1. Introduction

In the one curve approach to Ibor interest rate pricing, the Ibor fixing is link directly to the discounting curve. For an Ibor fixing in t_0 related to the period $[t_1, t_2]$, the link is

$$I_{t_0} = \frac{1}{\delta}\left(\frac{P(t_0,t_1)}{P(t_0,t_2)} - 1\right)$$

The framework and the relation are based on a risk free curve and the hypothesis that the deposits underlying the different Ibor indexes are risk free.

With the recent crisis it became more apparent that the hypothesis was not realistic and that a different approach was necessary. That necessity was already indicated well before the crisis in some literature and by some practitioners. A first step, splitting (risk-free) discounting and Ibor fixing, was proposed in a simplified set-up in Henrard (2007). The framework was later extended to a more flexible set-up in Henrard (2010) and is now the base for most of the multi-curves developments. Earlier developments like Tuckman and Porfirio (2003) and Boenkost and Schmidt (2004) had pointed to the weakness of the then framework but without providing a theoretically sound alternative. Further developments have been done in different directions, in particular in Kijima et al. (2009), Ametrano and Bianchetti (2009), Chibane and Sheldon (2009), Mercurio (2009), Morini (2009), Bianchetti (2010), Piterbarg (2010), Moreni and Pallavicini (2010), Pallavicini and Tarenghi (2010), Mercurio (2010a).

In the one curve approach, the pricing of interest rate futures, one of the most liquid interest rate products, has attracted a lot of attention. As a small list of related literature, we indicate: Kirikos and Novak (1997), Cakici and Zhu (2001),Piterbarg and Renedo (2004), Henrard (2005), Jäckel and Kawai (2005). Obviously one would like to price the futures also in the multi-curves framework. This was done for the one-factor Gaussian HJM model in Henrard (2010) using hypotheses on the discounting/forward spread and in Mercurio (2010b) in the LMM with stochastic basis.

The starting point of the multi-curves framework is the existence of an asset which is not directly related to the discount bonds: the Ibor coupon; this is the rational behind the hypothesis **L** in Henrard (2010). This is why we refer here to that framework as the *coupon multi-curves framework*. From the existence hypothesis, one *define* a function $P^j(s,t)$ in such a way that the usual project-and-discount formulas used in swap pricing are still valid. This way to proceed is purely conventional. It is also quite practical as all standard instruments can be priced with formulas similar to the one we are used to. The approach is purely based on that definition and selected to used the good old formulas; there is no deep fundamental reason behind it.

Date: First version: 30 April 2012; this version: 8 May 2012.
Version 1.1.

Electronic copy available at: http://ssrn.com/abstract=2053657

Could we chose a different definition and have a similarly nice framework? The answer is *Yes*, and this is what this note is tempting to describe. As our alternative framework is based on futures, we call it the *futures multi-curves framework*. Is this a new framework that one should switch to immediately? The answer is probably *No*. Nevertheless there are some possible future state of the universe where this may be useful. A perfect hedge should work in all states of the universe and a good risk manager should analyse potential scenarios. At least two good reasons to glance at the next section.

Cette histoire est vraie puisque je l'ai inventée...

Boris Vian

2. Where futures are not convex

An (interest rate) futures is characterized by a fixing date, denoted t_0, and an underlying j-Ibor rate. The rate fixing in t_0 is denoted $I^j_{t_0}$. This is the rate valid for the period $[t_1, t_2]$ with $t_1 = \text{Spot}(t_0)$ and $t_2 = t_1 + j$ months with the dates computed using the relevant conventions.

The fixing futures price in t_0 is $\Phi^j_{t_0}(t_0) = 1 - I^j_{t_0}$. This is the link between the futures and the underlying Ibor. The margining process for interest rate futures works in the following way. For a given *closing price* (as published by the exchange), the daily margin paid is that price minus the *reference price* multiplied by the notional and by the accrual factor of the futures. The reference price is the trade price on the trade date and the previous closing price on the subsequent dates.

Our existence hypothesis for the Ibor futures linked them to futures price processes in the sense of Hunt and Kennedy (2004).

I: The prices of the (j-Ibor) futures are futures price processes for each fixing date. The prices are continuous functions of time.

Once we have assumed that the instrument exists in our economy, we can give its price a name. We do it indirectly through the curves P^j.

Definition 1. *The forward curve P^j is the continuous function such that, $P^j(t,t) = 1$, $P^j(t,s)$ is an arbitrary function for $t \leq s < \text{Spot}(t) + j$, and for $t_0 \geq t$, $t_1 = \text{Spot}(t_0)$ and $t_2 = t_1 + j$*

$$\Phi^j_t(t_0) = 1 - \frac{1}{\delta}\left(\frac{P^j(t,t_1)}{P^j(t,t_2)} - 1\right). \tag{1}$$

The futures price is obtained directly from the forward curves (or more exactly the curve is obtained directly from the futures prices) without convexity adjustment. This definition justifies the note title: *My future is not convex*. A more explicit title would be: *With a different definition of the forward curve, the pricing of futures does not require a model adjustment from the forward discount factors ratio*; but this would be less cryptic.

With that definition, the link between $I^j_{t_0}$ and P^j is

$$I^j_{t_0} = \frac{1}{\delta}\left(\frac{P^j(t_0,t_1)}{P^j(t_0,t_2)} - 1\right).$$

In hypothesis **I** and Definition 1, we suppose the existence of a continuum of futures with all possible fixing date t_0. Obviously finance is discreet in payment dates, with at most one payment by day, and any real number t exists in practice only on discreet daily points. For futures there is the extra constraint that futures are traded with settlement date only every months on the short part of the curve and quarterly up to 10 years. This may appear as a lot less points than the usual FRAs and swaps. This is not really the case as the FRA are not specially liquid and mainly traded only on the short part with monthly maturities and above two year, there are only swap with annual maturities. The futures curve contains more points in the 2 to 10 years range. The coupon curves contains more points only in theory, not in practice.

Like in the coupon framework, we define a new variable:

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Definition 2. *The variable $\beta_t^j(t_1,t_2)$ is defined as a ratio of discount factors ratios*

$$\beta_t^j(t_1,t_2) = \frac{P^j(t,t_1)}{P^j(t,t_2)}\frac{P^D(t,t_2)}{P^D(t,t_1)}. \tag{2}$$

In the coupon framework an hypothesis often used is that the ratios β_t^j are constant through time. Can we use the same hypothesis here? The answer is no. By construction, Φ_t^j, which is a futures price process, is a N-martingale, with N_t the cash account. Consequently the ratio of projecting discount factors $P^j(t,t_1)/P^j(t,t_2)$ is also a N-martingale. If β_t^j is constant then the ratio $P^D(t,t_1)/P^D(t,t_2)$ is a N-martingale. On the other side, the same ratio is the ratio of an asset and the numeraire and is thus a $P^D(.,t_2)$-martingale. When rates are not deterministic, a constant β hypothesis would lead to a contradiction.

We propose to use the next best thing: a deterministic spread hypothesis

SD: The multiplicative coefficients between discount factor ratios, $\beta_t^j(t_1,t_2)$, defined in Equation (2), are deterministic for all t_1.

This is the equivalent to the constant spread hypothesis **S0** used in the coupon multi-curves framework.

An Ibor coupon pays the amount $\delta I_{t_0}^j$ in t_2. Its today's value is given by the following theorem.

Theorem 1. *In the futures multi-curves framework, under the hypothesis **I** and **SD**, the price of the Ibor coupon fixing in t_0 for the period $[t_1,t_2]$ is given by*

$$P^D(0,t_1)\beta_{t_0}^j(t_1,t_2) - P^D(0,t_2).$$

Proof. The proof is immediate. It is enough to use the link between $I_{t_0}^j$ and P^j, use the definition of $\beta_{t_0}^j$ and take the expectation with $P^D(.,t_2)$ as numeraire. □

The formula is very closed to the one for Ibor coupon in the coupon multi-curves framework. The difference is that here the β^j is taken in t_0, not in 0. We have shown above that the quantity can not be constant, so the two formulas are different and this is to be expected as the definitions of P^j are different.

It was proved in Henrard (2010) that the quantity $\gamma(t)P^D(t,t_1)/P^D(t,t_2)$ is a N-martingale in the one-factor Gaussian HJM model for

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2)-\nu(s,t_1))ds\right)$$

and ν the bond volatility in the one-factor Gaussian HJM model. This is the base of the pricing of futures in the coupon framework. Our next hypothesis is coherent with that observation

HJM1: The quantities $\beta_t^j(t_1,t_2)$ are such that

$$\beta_t^j = \beta_0^j\frac{\gamma(t)}{\gamma(0)}.$$

Under **HJM1** hypothesis, we have the following equalities

$$\frac{P^j(t,t_1)}{P^j(t,t_2)} = \frac{P^D(t,t_1)}{P^D(t,t_2)}\beta_t^i = \frac{P^D(t,t_1)}{P^D(t,t_2)}\gamma(t)\frac{\beta_0^i}{\gamma(0)}$$

with the first and the last variables N-martingales in the one factor Gaussian HJM model.

Theorem 2. *In the futures multi-curves framework, under the hypothesis **I** and **HJM1**, in the one-factor Gaussian HJM model the price of the Ibor coupon fixing in t_0 for the period $[t_1,t_2]$ is given by*

$$P^D(0,t_2)\frac{\beta_0^j(t_1,t_2)}{\gamma(0)} - P^D(0,t_2).$$

The convexity adjustment is now done on the Ibor coupon, not on the futures anymore. Note that the adjustment is obtained by dividing by the coefficient $\gamma(0)$ and not multiplying by it. Should the adjustment be called a *concavity adjustment*?

In this new framework, with the same hypothesis, on can also price a FRA:

Theorem 3. *In the futures multi-curves framework, under the hypothesis **I** and **HJM1**, in the one-factor Gaussian HJM model the price of the FRA fixing in t_0 for the period $[t_1, t_2]$ and settling in t_1 is given by*

$$P^D(0,t_1)\frac{(1-\gamma(0))+\delta(F_0^j-\gamma(0)K)}{1+\delta F_0^j}$$

3. And now what?

The hypothesis **HJM1** may seams a very strong assumption; it contains very specific deterministic hypothesis on the spread between discounting and forward curves and a specific HJM model for the curve dynamic.

Those hypothesis are exactly the equivalent of the one done currently for the forward curve constructions. If FRA's are used in the curve construction, the constant spread hypothesis **S0** of Henrard (2010) is implicitly assumed. If futures are used in the curve construction, a term structure hypothesis is implicitly assumed to be able to compute the convexity adjustment. The most used one is the one-factor Gaussian HJM model. It could be replaced by another one by adjusting the hypothesis **HJM1** accordingly. Even if the hypothesis seems very strong there are not different to the one used today in practice.

In the previous section we proposed a multi-curve framework based on exchange traded instruments with daily margining (interest rate futures) instead of OTC derivatives (swaps). Within the relatively reduce scope discussed here (interest rate futures and interest rate swaps) the framework is quite symmetrical with the current coupon multi-curves framework. Here the futures price is direct and the swaps need to be adjusted while in the standard framework it is the opposite.

Given that the current market is more centered around OTC swaps than around futures, it make sense to continue with the current framework. Nevertheless there are two trends that may lead to a change. There is a clear pressure by regulators to move part of the interbank transaction to exchange traded instruments. This could mean an increase in futures liquidity relative to FRA/swaps and maybe at some stage an increase in the futures offer (more serial futures, more one month futures, better offer on the futures packs).

On the other side, in the OTC market, there is a push for more standardization of the products but also of the legal terms (CSA in particular). One particular discussion is around the changes of the CSA terms and the collateral renumeration in particular. With the current terms often an overnight rate (fed fund; Eonia, etc.) is paid and the paying party has an embedded currency option. One potential solution to simplify the term of the CSA, which has been proposed by several market participants, would be to pay 0 interest on the collateral. This is equivalent to a futures margining. If that proposal, which simplify a certain number of practical problems, is put in place, what we call the futures multi-curve framework would be the natural one for the swaps with 0 rate collateral. The swaps without collateral would be an exception and priced using the adjustments proposed in Theorem 2. This is not an impossible scenario.

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