MULTIPLE CURVE CONSTRUCTION

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1. Introduction

In the post-credit-crunch world, swaps are generally collateralized under a ISDA Master Agreement (Andersen and Piterbarg p266), with collateral rates being Fed Funds (USD), Eonia (Euro), Sonia (GBP), etc.. This divorces the forward Libor rates from the discount factors, meaning two curves are needed to price swaps: a (risk-free) discount curve which is used to discount future cash flows, and a (risky) index (or forward) curve from which the forward Libor rates are derived. Further to this, there exists tenor swaps which pay a particular Libor tenor (e.g. 3M Libor every 3 months) plus a spread, and receive a different Libor tenor (e.g. 6M Libor every six months). Generally, a positive spread is paid on the leg of the shorter tenor, reflecting the greater risk of the longer tenor. So for a particular country a whole family of curves must be created - this could be expressed as a risk-free curve (e.g. OIS for USD), and a set of spread curves, one for each Libor tenor. Each country would need its own set of such curves.

Uncollateralized interest rate derivatives have credit risk, and proper treatment requires a credit valuation adjustment (CVA); something that we do not consider here.

2. Notation

- $P^{X,I}(t,T)$ The (pseudo) discount factor from time T to t for currency X and index I - if the index superscript is absent we have the risk-free curve. $P^{X,I}(T) \equiv P^{X,I}(0,T)$ is today's discount curve. For example $P^{\$,3M}(T)$ is the USD 3M Libor curve.
- $R^{X,I}(t,T) \equiv \frac{-1}{T-t}\ln(P^{X,I}(t,T))$ is the yield curve at time t.
- $L_i^{X,I}(t) \equiv L^{X,I}(t, S_i^{X,I}, T_i^{X,I}) \equiv \mathbb{E}_t^{X,T_i}[L^{X,I}(t, S_i^{X,I}, T_i^{X,I})]$ is the forward index (e.g. Libor) rate, where the expectation is taken at time t in the T_i forward measure (i.e. the numeraire is $P^X(t,T_i)$). The times $S_i^{X,I}$ and $T_i^{X,I}$ are the fixing (effective) and maturity times of the index.
- $t_1^X \dots t_M^X$ is the set of fixed payment times
- $t_1^{X,I} \dots t_N^{X,I}$ is the set of floating payment times
- τ_i^X is the year fraction for the fixed payment at time t_i^X
- $\alpha_i^{X,I}$ is the year fraction for the index rate $L_i^{X,I}$
- N_X an amount in currency X

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- $f_{X,Y}(t)$ the spot exchange rate at time t - amount of currency X per unit of currency Y. So $N_X = f_{X,Y} N_Y$ and $f_{X,Y} = (f_{Y,X})^{-1}$
- $f_{X,Y}(t,T)$ forward exchange rate seen at time t for exchange at time T

When the currency and/or index is clear we will suppress these superscripts.

3. Instruments

We now present how various Interest Rate Derivatives (IRD) can be priced using multiple curves. The goal, of course, is to construct multiple curves from market prices[1] of IRD. The companion document *The Analytic Framework for Implying Yield Curves from Market Data* discusses this in detail. Essentially, yield curves are represented as splines (and the discount factors trivially derived from these), with nodes (or knots) at the maturities of IRD. If the total number of nodes (across all relevant curves) equals the total number of IRD sensitive to these curves (i.e. have prices that depend on the curves), then it is just a multi-dimensional root-finding excise to find the values of the nodes that price the IRD back to the market exactly.

3.1. Interest Rate Swaps (IRS).

The forward index is related to the index curve (pseudo discount curve) by

$$L_i^{X,I}(t) = \frac{1}{\alpha_i^{X,I}}\left(\frac{P^{X,I}(t,S_i^{X,I})}{P^{X,I}(t,T_i^{X,I})} - 1\right) \tag{1}$$

This allows us to write down the floating leg reference index I in currency X as

$$\begin{aligned} PV^{X,I}(float) &= \sum_{i=1}^{N} \alpha_i^{X,I} L_i^{X,I} P^X(t,t_i^{X,I}) \\ &= \sum_{i=1}^{N}\left(\frac{P^{X,I}(t,S_i^{X,I})}{P^{X,I}(t,T_i^{X,I})} - 1\right) P^X(t,t_i^{X,I}) \end{aligned} \tag{2}$$

and fixed leg of a swap in currency X as

$$PV^X(fixed) = k\sum_{i=1}^{M} \tau_i^X P^X(t,t_i^X) \tag{3}$$

Here the fact the the floating leg depends on two curves is clear. Academic treatment normally has $T_i^{X,I} = S_{i-1}^{X,I}$ (i.e. spanning forward rates) and $T_i^{X,I} = t_i^{X,I}$ (i.e. the floating payment is on the same date as the maturity of the forward rate). Neither of these conditions necessarily holds in practice (they will differ by a few days), so three sets of dates will be needed for the floating leg.

The PV of a payer swap (i.e. pay fixed and receive floating payments based on the index I, in currency X) is

$$PV^{X,I}(t) = \sum_{i=1}^{N}\left(\frac{P^{X,I}(t,S_i^{X,I})}{P^{X,I}(t,T_i^{X,I})} - 1\right) P^X(t,t_i^{X,I}) - k\sum_{i=1}^{M}\tau_i^X P^X(t,t_i^X) \tag{4}$$

[1] price in a generic sense - in practice it could be swap rates, spreads, etc. that make an instrument have zero PV

so the par-swap rate is given by

$$k = \frac{\sum_{i=1}^{N}\left(\frac{P^{X,I}(t,S_i^{X,I})}{P^{X,I}(t,T_i^{X,I})} - 1\right) P^X(t,t_i^{X,I})}{\sum_{i=1}^{M} \tau_i^X P^X(t,t_i^X)} \tag{5}$$

The market will give a set of swap rates for different length swaps (typically 2 to 30 years). Without making some further assumptions (e.g. constant spread), it is not possible to back out both the discount and index curves just from the set of swaps. However, if the discount curve were extraneously given, then a index curve with the same number of nodes as swap rates is easily found[2].

3.2. **OIS Swaps.** In the USD market, from one week to twelve months, an OIS swap has a single floating payment equal to the geometric average of the Fed Funds overnight rate. That is,

$$\text{float leg} = \prod_{i=1}^{N}\left(1 + \frac{r_i}{N_{year}}\right) \tag{6}$$

where r_i is the overnight rate on day i, N is the number of days in the period, and N_{year} is the number of days in a year. This can be approximated by a continuously compounded rate

$$\text{float leg} = \exp\left(\int_{t_0}^{t_1} r_s ds\right) \tag{7}$$

with t_0 being the trade date (normally 1 or 2 days from now) and t_1 being the payment date. The fixed leg is just $1 + \alpha k$ where α is the year fraction. This means that the PV of this swap is

$$\begin{aligned} PV =& \mathbb{E}\left[e^{-\int_0^{t_1} r_s ds}\left(e^{\int_{t_0}^{t_1} r_s ds} - (1+\alpha k)\right)\right] \\ =& \mathbb{E}\left[e^{-\int_0^{t_0} r_s ds}\right] - (1+\alpha k)\mathbb{E}\left[e^{-\int_0^{t_1} r_s ds}\right] \end{aligned} \tag{8}$$

where the risk-free rate used for discounting is identical to the overnight rate - the OIS curve is the risk-free curve for USD (and other markets where similar instruments exist). This can be written in terms of zero coupon bounds, $P(0,t)$ as

$$PV = P(0,t_0) - (1+\alpha k)(P(0,t_1) \tag{9}$$

which for zero PV gives

$$k = \frac{1}{\alpha}\left(\frac{P(0,t_0)}{P(0,t_1)} - 1\right) \tag{10}$$

which of course is exactly the form for Libor rates. Therefore, out to one year we can price OIS swaps using a single (risk-free) curve, and by inversion we can construct the OIS curve (to one year) from the OIS swap rate.

[2]The reverse does not necessarily hold, as the payment netting means there is only weak sensitivity of the swap rate (or equivalently the PV) to the discount curve.

For two to ten years, OIS swaps have annual coupons and can be handled exactly like ordinary fixed-float IRS where the index and discount curves are both the OIS (risk-free) curve. Beyond 10 years in the USD market we must consider basis swaps (see next section).

In the Euro and GBP markets, EONIA and SONIA swaps go out to 30 years, which allows construction of full risk-free curves in these currencies. The JPY market only has OIS swaps to 3 years, and cross-currency swaps (CCS) are needed to construct the full OIS curve (see section 3.5). This is also the case in other OECD countries.

3.3. **Basis Swap in a Single Currency.** A basis swap exchanges payments based on one index (e.g. Fed-Funds) for another (e.g. 3m-Libor) on the same notional amount. A tenor swap exchanges payments based on different Libor tenors, e.g. 3 month Libor paid quarterly for 6 month Libor paid semi-annually.

If legs of the swap (A & B) have reference rates, $L^{X,A}$ and $L^{X,B}$, the PV of the receiver of leg A is

$$
\begin{aligned}
PV^X &= \sum_{i=1}^{N_A} \alpha_i^{X,A} L_i^{X,A} P^X(t, t_i^{X,A}) - \sum_{i=1}^{N_B} \alpha_i^{X,B} (L_i^{X,B} + s) P^X(t, t_i^{X,B}) \\
&= \sum_{i=1}^{N_A} \left(\frac{P^{X,A}(t, S_i^{X,A})}{P^{X,A}(t, T_i^{X,A})} - 1 \right) P^X(t, t_i^{X,A}) \\
&\quad - \sum_{i=1}^{N_B} \left(\frac{P^{X,B}(t, S_i^{X,B})}{P^{X,B}(t, T_i^{X,B})} - 1 + \alpha_i^{X,B} s \right) P^X(t, t_i^{X,B})
\end{aligned} \tag{11}
$$

where s is the spread, which in this set up can take positive or negative values. It is clear that the PV depends on the value of three curves. Making the PV zero, the spread becomes

$$
s = \frac{\sum_{i=1}^{N_A} \alpha_i^{X,A} L_i^{X,A} P^X(t, t_i^{X,A}) - \sum_{i=1}^{N_B} \alpha_i^{X,B} L_i^{X,B} P^X(t, t_i^{X,B})}{\sum_{i=1}^{N_B} \alpha_i^{X,B} P^X(t, t_i^{X,B}))} \tag{12}
$$

If two of the curves are already known, these spreads can be used to construct the third. For example, if the discount curve (from OIS swaps) and the 3M Libor curve (from Libor rates, FRAs and IRS) are known, the 6M Libor curve can be found from 3M-6M tenor swap spreads.

The interesting thing for the USD market is using the US Fed Funds spreads - the USD Basis Swap Fed Funds versus USD three-month Libor - to extend the OIS curve to 30 years. If the Fed Funds leg were just the geometric average of the FF effective rate (as is the case for OIS), then there would be just two curves involved - the OIS curve (for discounting and calculating the FF-based floating payments) and the 3M Libor curve. Since these are also the two curves involved in pricing USD IRS, the two sets of market information (US Fed Funds spreads and USD swap rates) can be used to simultaneously construct the OIS and 3M Libor curves out to 30 years.

Unfortunately, the FF basis swap cash flows are calculated from an arithmetic average of the effective rate. The payment can be written as

$$\frac{\alpha}{N}\sum_{i=1}^{N} r_i \tag{13}$$

where α is the year fraction and the index runs of the days in the period. The expected PV of this payment can be approximated by

$$L^{\$,FF} = \mathbb{E}\left[e^{-\int_0^{t_{i+1}} r_s ds} \int_{t_i}^{t_{i+1}} r_s ds\right] \tag{14}$$

with r_s being the (risk-free) short rate. If r_s is high in the observation period between t_i and t_{i+1} then the payment at t_{i+1} will be high, but the discounting will be greater lowering the overall effect on the PV of the payment (the same argument holds if r_s is low in that period). The simplest approach is pretend the the averaging is geometric and proceed as for OIS. A more correct approach would be to express equation 14 in terms of discount factors and a convexity adjustment.

3.4. **Forward FX rates.** The forward FX rate is related to the spot and the discount curves in each currency by

$$f_{X,Y}(t,T) = f_{X,Y}(t)\frac{P^Y(t,T)}{P^X(t,T)} \tag{15}$$

These instruments are liquid in a range of currency pairs out to a few years, allowing the short end of a discount curve in one currency to be found from the short end in another[3].

3.5. **Cross-Currency Swaps (CCS).** A CCS is effectively an exchange of a FRN with notional N_X in currency X for one with notional N_Y in currency Y. On the trade date (typically a couple of days forward) the notional amounts are exchanged. Coupon payments are then made at defined dates, with the reverse exchange of notionals at maturity.

3.5.1. *Fixed-Fixed CCS.* For a fixed-for-fixed CCS the PV from a currency X investor is

(16)

$$\begin{aligned} PV =& N_x \left(P^X(0,T) + P^X(0,T) + k_X \sum_{i=1}^{N_X} \tau_i^X P^X(0,t_i^X) \right) \\ & - N_Y \left(-P^X(0,t_0) f_{X,Y}(0,t_0) + P^X(0,T) f_{X,Y}(0,T) + k_Y \sum_{i=1}^{N_Y} \tau_i^Y P^X(0,t_i^Y) f_{X,Y}(0,t_i^Y) \right) \end{aligned}$$

Here coupon payments can be at different times on each leg, but the initial and final (reverse) exchange of notional are at times t_0 and T. The NPV of the initial exchange is

$$NPV = -N_x P^X(0,t_0) + N_Y P^X(0,t_0) f_{X,Y}(0,t_0) \tag{17}$$

[3]This doesn't require root finding - the node values can just be read off.

which can be made zero by setting $N_Y = N_X f_{X,Y}(0,t_0)$. It can be the case that $N_Y = N_X f_{X,Y}(0)$ which means there could be a small initial payment if the (two day) forward FX rate differs from spot. Assuming the former condition holds, we rewrite the PV as (taking $N_X = 1$)
(18)

$$PV = P^X(0,T) + k_X \sum_{i=1}^{N_X} \tau_i^X P^X(0,t_i^X) - f_{X,Y}(0)\left(P^Y(0,T) + k_Y \sum_{i=1}^{N_Y} \tau_i^Y P^Y(0,t_i^Y)\right)$$

If k_X is set to make the X currency payments PV to zero (i.e. $k_X = \frac{P^X(0,t_0)-P^X(0,T)}{\sum_{i=1}^{N_X}\tau_i^X P^X(0,t_i^X)}$), then the PV of the whole swap becomes

$$PV = P^X(0,t_0) - f_{X,Y}(0)\left(P^Y(0,T) + k_Y \sum_{i=1}^{N_Y} \tau_i^Y P^Y(0,t_i^Y)\right) \tag{19}$$

So for a fair swap we must set

$$k_Y = \frac{f_{X,Y}(0)P^Y(0,T) - P^X(0,t_0)}{\sum_{i=1}^{N_Y}\tau_i^Y P^Y(0,t_i^Y)} \tag{20}$$

If a set of fixed coupons k_Y are given then the currency Y discount curve can be constructed.

3.6. **Float-Float CCS.** This exchanges floating payments based on an index in one currency for floating payments based on an index in a different currency. Assuming the same treatment of notional payments are above (i.e. they net to zero), then PV of this swap from the point of view of the X investor is

$$\begin{aligned} PV =& P^X(0,T) + \sum_{i=1}^{N_X} \alpha_i^{X,A} L_i^{X,A} P^X(0,t_i^{X,A}) \\ &- f_{X,Y}(0)\left(P^Y(0,T) + \sum_{i=1}^{N_Y} \alpha_i^{Y,B}(L^{Y,B}+s)P^Y(0,t_i^{Y,B})\right) \end{aligned} \tag{21}$$

The spread s for a fair swap is
(22)

$$s = \frac{P^X(0,T) - f_{X,Y}(0)P^Y(0,T) + \sum_{i=1}^{N_X}\alpha_i^{X,A}L_i^{X,A}P^X(0,t_i^{X,A}) - f_{X,Y}(0)\sum_{i=1}^{N_Y}\alpha_i^{Y,B}L^{Y,B}P^Y(0,t_i^{Y,B})}{\sum_{i=1}^{N_Y}\alpha_i^{Y,B}P^Y(0,t_i^{Y,B})}$$

The market values (PV or spread) depends on four curves - the discount curve in each currency and the index curve in each currency.

As a concrete example, assume we have already constructed the USD discount curve and 3M Libor curve (from OIS swaps, Fed Funds basis swaps, Libor rates, FRAs and swaps). The short end of the JPY discount curve can be covered by OIS swaps (which go out to 3 years in JPY), and the short end of the Libor curve by Libor rate and FRA (out to 2 years). However there is no OIS market beyond 3 years, and this is where USDJPY CCS spreads (swapping 3M USD Libor for 3M JPY Libor) and JPY IRS (swapping fixed for 3M Libor in the JPY domestic market) come in. Using these (together with the known USD curves) we can construct the JPY discount curve and 3M Libor curve out to 30 years.

There is an additional complication to the above story: The liquid USDJPY CCS market is in 3M Libor, but the main JPY IRS market is in 6M Libor. However there is a JPY 3M/6M tenor swap. So we will need to fit 3 curves simultaneously using the USD curves together with OIS swaps, FRA (6M), IRS (6M tenor), USDJPY CCS and 3M/6M JPY tenor swaps.

4. Spread Representation

Variations of the USD 3M Libor curve from the discount (OIS) curve will generally be small. If we treat the discount curve (in each currency) as the fundamental curve, then all of the various index curves can be represented as spread curves over this, i.e.

$$P^{X,I}(t,T) = Q^{X,I}(t,T)P^{X}(t,T) \tag{23}$$

or, in terms of the yield curves

$$R^{X,I}(t,T) = S^{X,I}(t,T) + R^{X}(t,T) \tag{24}$$

where $S^{X,I}(t,T) = \frac{-1}{T-t}\ln(Q^{X,I}(t,T))$. The spread curves could then be endowed with far fewer nodes than the fundamental curve, while the index curves still display complex structure (inherited from the fundamental curve).

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