INTEREST RATE FUTURES AND THEIR OPTIONS: SOME PRICING APPROACHES

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Abstract. Exchange-traded interest rate futures and their options are described. The future options include those paying an up-front premium and those with future-like daily margining.

1. Introduction

The pricing is done in a multi-curve framework, with discounting curve denoted

$$P^D(s,t).$$

and forward or estimation curves denoted

$$P^j(s,t)$$

with j the tenor of the relevant Ibor index or $j = O$ for the overnight index. Most of the futures are linked to three month Ibor indices, but some are linked to the one month Ibor index or an overnight index.

The (Ibor or overnight) fixing taking place on t_0 for the period $[t_1; t_2]$ with accrual factor δ is linked to the curve by

$$1 + \delta L^j_{t_0} = \frac{P^j(t_0,t_1)}{P^j(t_0,t_2)}.$$

2. Ibor-like Interest Rate Futures

The futures type described here are the Ibor futures as traded on CME[1] for USD and JPY, on NYSE-Euronext-Liffe[2] for EUR, GBP, CHF and USD and on EUREX[3] for EUR. The dates related to those futures are based on the third Wednesday of the month[4], which is the *start date* of the Ibor rate underlying the future.

The rate is fixed at a *spot lag* prior to that date; the fixings usually take place on the Monday or on the Wednesday itself. The fixing date is also the *last trading date* for the future. The *end date* of the Ibor rate period is three months[5] after the *start date* (using the conventions associated to the relevant Ibor-index).

The margining process works in the following way. For a given *closing price* (as published by the exchange), the daily margin paid is that price minus the *reference price* multiplied by the notional and by the accrual factor of the future. Equivalently, it is the price difference multiplied by one hundred and by the *point value*, the point value being the margin associated with a one (percentage) point change in the price, the reference price the trade price on the trade date, or the previous closing price on subsequent dates.

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[1] www.cmegroup.com
[2] www.euronext.com
[3] www.eurexchange.com
[4] When the day not a business day, it is adjusted to the following day.
[5] Futures on the one month Ibor also exist, those on three month have higher volume.

The price of the future on t is denoted Φ_t^j. On the fixing date at the moment of the publication of the Ibor rates L_t^j, the future price is $\Phi_t^j = 1 - L_t^j$. Before that moment, the price evolves with supply and demand.

For three month futures, the nominal is 1,000,000 (500,000 for GBP and 100,000,000 for JPY) and the accrual factor is 1/4. For one month futures, the nominal is 3,000,000 and the accrual factor is 1/12. In both cases, the nominal multiplied by the accrual factor is the equal (250,000 for USD, 125,000 for GBP and 25,000,000 for JPY). The value of one (percentage) point is 2,500 currency units for USD, 1,250 for GBP and 250,000 for JPY.

The *tick value* is the value of the smallest permissible increment in price. The prices usually change in 1.0 or 0.5 basis point increments.

The futures are designated by character codes. The first part depends on the data provider and is usually two or four characters. The main codes are given in Table 1. The second part describes the month, with the code given in Table 2, and the year by its last digit. As interest rate futures are only quoted up to 10 years in the future, there is no ambiguity in using only one figure for the year. Note also that it means that when a future reaches its last trading date, a new one is created a couple of days later with the same name but for a maturity 10Y in the future.

Currency	Tenor	Exchange	Underlying	Notional
USD	3M	CME	LIBOR	1,000,000
USD	1M	CME	LIBOR	3,000,000
EUR	3M	Eurex	EURIBOR	1,000,000
EUR	3M	Liffe	EURIBOR	1,000,000
GBP	3M	Liffe	LIBOR	1,000,000
CHF	3M	Liffe	LIBOR	1,000,000
JPY	3M	SGX/CME	TIBOR	1,000,000
JPY	3M	SGX	LIBOR	1,000,000

TABLE 1. Futures details and codes.

Month	Code	Month	Code	Month	Code
January	F	February	G	March	H
April	J	May	K	June	M
July	N	August	Q	September	U
October	V	November	X	December	Z

TABLE 2. Rate futures month codes.

The future fixing date is denoted t_0. The fixing is on the Ibor rate between $t_1 = \text{Spot}(t_0)$ and $t_2 = t_1 + j$ month and the accrual factor for the period $[t_1, t_2]$ is denoted δ.

The generic future price process theorem (see [Hunt and Kennedy, 2004, Theorem 12.6]), states that, in the cash account numeraire with probability $\mathbb{N}$, the future price is given by

$$\Phi_t = \mathrm{E}^{\mathbb{N}}\left[\Phi_{t_0} | \mathcal{F}_t\right]. \tag{1}$$

2.1. **Pricing by discounting.** In this approach the margining details of the future are not taken into account. The future is priced like an Ibor coupon. The future rate is computed as the forward rate

$$F_0^j = \frac{1}{\delta}\left(\frac{P^j(0,t_1)}{P^j(0,t_2)} - 1\right).$$

The future price is

$$\Phi_0^j = 1 - F_0^j.$$

2.2. **Pricing in the Hull-White one factor model.** The model used is a Hull-White one-factor model, or extended Vasicek model. The multi-curve pricing formula presented here is from Henrard [2010]. It uses the deterministic spread hypothesis between the curves. The model volatility is $\sigma(t,s)$ and we denote $\nu(t,u) = \int_t^u \sigma(t,s)ds$.

Theorem 1. *Let $0 \leq t \leq t_0 \leq t_1 \leq t_2$. In the HJM one-factor model on the discount curve in the multi-curve framework and with the independence hypothesis between spread and discount factor ratio **SI** proposed in Henrard [2010], the price of the futures fixing at t_0 for the period $[t_1, t_2]$ with accrual factor δ is given by*

$$\begin{aligned} \Phi_t^j &= 1 - \frac{1}{\delta}\left(\frac{P^j(t,t_1)}{P^j(t,t_2)}\gamma(t) - 1\right) \\ &= 1 - \gamma(t)F_t^j + \frac{1}{\delta}(1-\gamma(t)) \end{aligned} \tag{2}$$

where

$$\gamma(t) = \exp\left(\int_t^{t_0} \nu(s,t_2)(\nu(s,t_2) - \nu(s,t_1))ds\right).$$

The *rate adjustment* $(\gamma(t)F_0^j - \frac{1}{\delta}(1-\gamma(t)) - F_0^j)$ is given in Figure 1 for several volatility levels and futures up to 5 years. The Hull-White volatility used are constant with level $\sigma = 0.005$, 0.01 and 0.02.

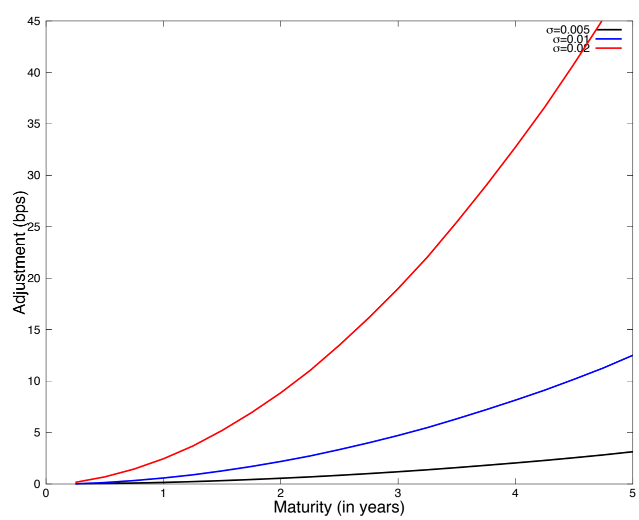

Figure 1. Rate adjustments for different level of volatilities and future up to 5 years.

3. Interest Rate Futures Options: Premium

An option on future is described by the underlying future, an option expiration date θ, a strike K and an option type (Put or Call). The expiration is before or on the future last trading date: $\theta < t_0$.

The option on futures dealt with in this section are American type and pay the premium up-front at the transaction date. There is no margining process for the option. This type of option is traded on the CME exchange for eurodollar futures (one and three months).

There are three type of options: quarterly options, serial options and mid-curve options. Quarterly options expire on the last trading date of the underlying future, i.e. $\theta = t_0$. Serial and mid-curve options expire before the future last trading date. For serial options, the delay is one or two months (plus one week-end). For mid-curve options, the delay is one, two or four years.

The quoted price for the options follow the same rule as the future. For a quoted price, the amount paid is the price multiplied by the notional and by the accrual factor of the underlying future.

4. Interest Rate Futures Options: Margin

An option on future is described by the underlying future, the *option expiration date* θ, the *strike* K and an option type (Put or Call). The expiration is before or on the future last trading date: $\theta \leq t_0$.

The option on futures dealt with in this section are American type and have a future-like margining process. This type of option is traded on the Liffe for EUR, GBP, CHF and USD futures (three months) and Eurex for EUR (three months).

Note that there are two margin processes involved in this instrument: one on the underlying future and one on the option itself.

The quoted price for the options follows the same rule as for the future. For a quoted price, the daily margin paid is the current closing price, minus the reference price, multiplied by the notional and the accrual factor of the underlying future. The reference price is the trade price on the trade date, and the previous closing price on subsequent dates.

Currency	Tenor	Exchange	Underlying	Type
USD	3M	LIFFE	LIBOR	Option on future
USD	3M	LIFFE	LIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	Option on future
EUR	3M	LIFFE	EURIBOR	Mid-Curve Options
EUR	3M	LIFFE	EURIBOR	2 year Mid-Curve Options
GBP	3M	LIFFE	LIBOR	Option on future
GBP	3M	LIFFE	LIBOR	Mid-Curve Options
GBP	3M	LIFFE	LIBOR	2 year Mid-Curve Options
CHF	3M	LIFFE	LIBOR	Option on future

Table 3. Interest rate futures options codes.

4.1. SABR on future rate.

The future price is a martingale in the cash-account numeraire. The *future rate* $R_t = 1 - \Phi_t$ is also a martingale.

Due to the margining process on the option, the price of the option with margining is (see [Hunt and Kennedy, 2004, Theorem 12.6])

$$\mathrm{E}^{\mathbb{N}}\left[(\Phi_\theta - K)^+\right] = \mathrm{E}^{\mathbb{N}}\left[((1-K) - R_\theta)^+\right].$$

The future rate can be modelled through any process that leads to a martingale. We choose to model it with a SABR process without drift.

Let $\mathrm{SABR}(S, M, T)$ be the price of an option of strike M and type T when the underlying is S.

In the SABR on future rate framework, the future option will be priced by

$$\mathrm{SABR}(1 - \Phi_0^j, 1 - K, !T)$$

where $!\,\mathrm{Call} = \mathrm{Put}$ and $!\,\mathrm{Put} = \mathrm{Call}$ (a call on the price is a put on the rate).

In this implementation, the future price is calculated without convexity adjustment.

4.2. **SABR on future rate: curve sensitivity.** The curve sensitivity is obtained from the above formula by composition. The derivative of the SABR formula with respect to the underlying is known and the derivative of the future price with respect to the rates is also known.

4.3. **American option.** The options on futures are actually American options, even if we have treated them as European above. For the explanation why for margined options on margined futures, there is no difference in valuation between American and European options (i.e. it is never optimal to exercise early), see Chen and Scott [1993]. The American option has the same price as the European option.

The argument proceeds as follows. Let $f(x)$ be the option pay-off function (i.e. $f(x) = (x-K)^+$ for a call and $f(x) = (K-x)^+$ for a put). Those two functions are convex. The Jensen's inequality can be applied

$$\mathrm{E}^{\mathbb{N}}\left[f(\Phi_\theta)\right] \geq f\left(\mathrm{E}^{\mathbb{N}}\left[\Phi_\theta\right]\right) = f(\Phi_0).$$

where the future price equation (1) and its martingale property is used in the last line. The left hand side of the inequality is the European option value now and the right hand side is the pay-off for early exercise. This proves that it is never optimal to exercise before expiry.

5. Fed fund futures

The *30-Day Federal Funds Futures* (simply called Fed Funds futures) are based on the monthly average of overnight Fed Funds rate for the contract month. The notional is 5,000,000 USD. The contract months are the first 36 calendar months. They are quoted on CBOT[6] for USD. In practice, even if the Ibor and Fed Fund curves are different, the technical adjustment are similar. In the later case the underlying is an arithmetic average and the adjustment has to be done several times (one for each day, on average).

Let $0 < t_0 < t_1 < \cdots < t_n < t_{n+1}$ be the relevant date for the Fed Funds futures, with t_1 the first business day of the reference month, t_{i+1} the business day following t_i and t_{n+1} the first business day of the following month. Let δ_i be the accrual factor between t_i and t_{i+1} ($1 \leq i \leq n$) and δ the accrual factor for the total period $[t_1, t_{n+1}]$. The day count convention for the USD overnight is ACT/360.

The overnight rates between t_i and t_{i+1} are given in t_i by F_i^O with

$$1 + \delta_i F_i^O = \frac{1}{P^O(t_i, t_{i+1})}.$$

The future price on the final settlement date t_{n+1} is

$$\Phi_{t_{n+1}} = 1 - \frac{1}{\delta}\left(\sum_{i=1}^{n} \delta_i F_i^O\right).$$

The margining is done on the price multiplied by the notional and divided by the one month accrual fraction (1/12).

The model used is a Hull-White one-factor model or extended Vasicek model. The result uses the deterministic spread hypothesis between the curves. The model volatility is $\sigma(t,s)$ and we denote $\nu(t,u) = \int_t^u \sigma(t,s)ds$.

Theorem 2. *In the HJM one-factor model on the discount curve in the multi-curve framework and with the deterministic hypothesis between spread and discount factor ratio* ***S0*** *proposed in Henrard [2010], the price of the average overnight future for the period* $[t_1, t_{n+1}]$ *is given for* $t_j \leq t < t_{j+1}$ *(*$j \in \{0, 1, \ldots, n\}$*) by*

$$\Phi_t = 1 - \frac{1}{\delta}\left(\sum_{i=1}^{j} \delta_i F_i^O + \sum_{i=j+1}^{n} \delta_i \left(\frac{P^O(t,t_i)}{P^O(t,t_{i+1})}\gamma_i - 1\right)\right)$$

[6] www.cmegroup.com

where

$$\gamma_i = \exp\left(\int_t^{t_i} \nu(s,t_{i+1})(\nu(s,t_{i+1}) - \nu(s,t_i))ds\right).$$

The formula is divided into two parts to cope with the case where the averaging period has started already. In those cases the convexity adjustments γ_i are very small.

6. Bank bill futures (AUD style)

AUD bill futures are physically settled. At expiry, different bills can be delivered. The bills eligible for delivery are bills with between 85 and 95 days to maturity at the settlement date. The issuers of the bills are the one in the eligible counterparties.

The party short of the future chooses the bill to be delivered[7]. The short party has an option and he will, of course, choose to deliver the cheapest bill. This is a situation similar to the one for bond futures in other currencies.

Let θ be the expiry (or announcement) date and t_0 be the settlement date. In practice, those dates are the second Friday of the future month and the next business day (Monday).

Let $t_i (1 \leq i \leq N)$ denote the possible maturity dates of the bills[8]. At settlement, the price received for the bill will depend of the last quoted future index that we denote F_θ. The yield associated to this index is $R_\theta = 1 - F_\theta$. The price paid is

$$\frac{1}{1 + \delta_i R_\theta}$$

where δ_i is the accrual factor associated to the dates t_0 and t_i. For AUD bill futures this factor is the number of calendar days between the two dates divided by 365. In exchange for the price the short party gives the bill with a notional equal to the notional of the future[9].

7. Implementation

The future security description is in the class `InterestRateFuture`.

The implementation of the future pricing by discounting is in the method `InterestRateFutureTransactionDiscountingMethod`. The implementation for the Hull-White one factor model is in the method `InterestRateFutureTransactionHullWhiteMethod`.

The implementation of the option on future with margin is in the method `InterestRateFutureOptionMarginTransactionSABRMethod`. The SABR parameters are represented by surfaces. The first dimension is the time to expiration; the second dimension is the delay between option expiry and future last trading date. In this implementation, the future price is computed without convexity adjustment (discounting method). This is sufficient for short dated futures but probably not for long-dated mid-curve options.

References

R. R. Chen and L. Scott. Pricing interest rate futures options with futures-style margining. *The Journal of Futures Market*, 13(5):15–22, 1993. 5

M. Henrard. The irony in the derivatives discounting part II: the crisis. *Wilmott Journal*, 2(6): 301–316, December 2010. Preprint available at SSRN: http://ssrn.com/abstract=1433022. 3, 5

[7] Actually, for each contract he can choose up to 10 different bills of AUD 100,000 each.

[8] In practice there are nine possible dates taking the week-end into account

[9] The notional of the bill futures is AUD 1,000,000. This notional can be split into several physical bills, up to 10 pieces of AUD 100,000.

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Marc Henrard, Quantitative Research, OpenGamma
E-mail address: quant@opengamma.com