

SWAP AND CAP/FLOORS WITH FIXING IN ARREARS OR PAYMENT DELAY

OPENGAMMA QUANTITATIVE RESEARCH



ABSTRACT. Pricing methods for swaps and caps/floors with fixing in arrears or payment delays are proposed. The methods are based on replication.

1. INTRODUCTION

Standard swaps and cap/floors on Ibor fix in advance (at the start of the relevant Ibor period) and pay in arrears (at the end of the period). The dates related to cap/floors are in line with the theoretical dates on which the Ibor rates are fixed. This is the case for standard cap/floor, (as for standard swaps), up to a couple of days due to business day adjustment conventions.

Other swaps and caps/floors fix at the end of the period¹: they are called *in arrears* swaps or cap/floors. Some other swaps or caps/floors fix on tenors that are not natural with respect to their payment frequency, e.g. a quarterly payment of a six month Ibor rate fixed at the start of the period. Those Ibor products have in common that the payment date is not natural for the deposit rate underlying the fixing.

This note proposes some approaches to price these products by replications. It is partly based on [Andersen and Piterbarg, 2010, Section 16.4]

The pricing is done in a multi-curve framework. The discounting curve is denoted

$$P^D(s, t).$$

The forward or estimation curves are denoted

$$P^j(s, t)$$

with j the tenor of the relevant Ibor index.

The Ibor fixing taking place in t_0 for the period $[t_1; t_2]$ with accrual factor δ is linked to the curve by

$$1 + \delta L_{t_0}^j = \frac{P^j(t_0, t_1)}{P^j(t_0, t_2)}.$$

The ratio between discounting and forward curves are given by

$$(1) \quad \beta_t^j(u, u + j) = \frac{P^j(t, u)}{P^j(t, u + j)} \frac{P^D(t, u + j)}{P^D(t, u)}.$$

As proposed in Henrard [2010], we use the simplifying deterministic hypothesis **S0** between the spread and discount factor ratios:

S0: The multiplicative coefficients between discount factor ratios, $\beta_t^j(u, u + j)$, defined in Equation (1), are constant through time: $\beta_t^j(u, u + j) = \beta_0^j(u, u + j)$ for all t and u .

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¹More precisely, they fix at a spot lag before the payment date.

2. STANDARD LIBOR

Let θ be the fixing date of an Ibor rate for the period $[t_0, t_1]$. The standard products pay in t_1 and their pay-offs are $p(L_\theta^j)$. For floating coupons, $p(x) = x$ and for caps, $p(x) = (x - K)^x$ with K the strike (the multiplication by the notional and the accrual factor are not taken into account).

In the $P^D(\cdot, t_1)$ numeraire, the price of the standard product (with payment in t_1 can be written as

$$V^{\text{Std}} = P^D(0, t_1) \mathbb{E}^{t_1} \left[p(L_\theta^j) \right].$$

As L_t^j is a martingale, the price of the coupon is $P^D(0, t_1)L_0^j$. The other type of pay-off T can be a Cap (Call) or a Floor (put). The generic price is

$$V^{\text{Std}} = P^D(0, t_1) \text{Base}(L_0^j, K; T)$$

where Base is some base formula for a forward L , a strike K , and a type T

In the Black-Scholes framework

$$\text{Base}(L, K; T) = \text{BS}(L, K, \sigma; T)$$

where

$$\text{BS}(L, K, \sigma; T) = \omega(LN(\omega d_1) - KN(\omega d_2))$$

In the SABR with implicit volatility, the same formula is used but with a strike and forward dependent volatility

$$\sigma = \sigma(L, K).$$

In the SABR approach with extrapolation for high strikes (see [OpenGamma Research \[2011\]](#) for OpenGamma implementation) the SABR base formula is used up to a cut-off strike above which a specific extrapolation is used.

The exact base pricing is not important for the pricing described below. The only important part is that there is a pricing mechanism for all standard products with any strike K and forward L :

$$\text{Std}(L, K, ; T) = P^D(0, t_1) \text{Base}(L_0^j, K; T).$$

3. IBOR-IN-ARREARS

In this section we suppose that the payment of an Ibor-related product (swap floating coupon or cap/floorlet) take place on t_0 and its pay-off is $p(L_\theta^j)$. For floating coupons, $p(x) = x$ and for caps, $p(x) = (x - K)^x$ with K the strike.

In the $P^D(\cdot, t_1)$ numeraire, the price of the (in arrears) product can be written as

$$V^{\text{IA}} = P^D(0, t_1) \mathbb{E}^{t_1} \left[\frac{P^D(\theta, t_0)}{P^D(\theta, t_1)} p(L_\theta^j) \right].$$

The ratio of discount factors can link the Libor rate through the factor β with

$$\frac{P^D(\theta, t_0)}{P^D(\theta, t_1)} = \frac{1}{\beta^j} (1 + \delta L_\theta^j).$$

The price of the in-arrears product is then

$$V^{\text{IA}} = P^D(0, t_1) \frac{1}{\beta^j} \mathbb{E}^{t_1} \left[(1 + \delta L_\theta^j) p(L_\theta^j) \right]$$

and only the rate L appears in the expectation. The price can be further written as

$$V^{\text{IA}} = \frac{1}{\beta} \left(V^{\text{std}} + \delta P^D(0, t_1) \mathbb{E}^{t_1} \left[L_\theta^j p(L_\theta^j) \right] \right).$$

The value of the expectation can be computed by standard replication arguments as described in [[Andersen and Piterbarg, 2010](#), Proposition 8.4.13] for smooth function or in [[Hagan, 2003](#), Equation 2.14] for option-like pay-off. We suppose that rates can only be positive and that the price of floor with strike 0 is 0.

3.1. Swap in arrears. For the swap, $p(x) = x$ and we apply the above proposition with $f(x) = x^2$ and $K^* = 0$. The expectation is then

$$\mathbb{E}^{t_1} \left[L_\theta^j p(L_\theta^j) \right] = 2 \int_0^\infty \text{Base}(L_0^j, k; \text{Call}) dk.$$

The price is thus

$$V^{\text{IA}} = \frac{1}{\beta} \left(P^D(0, t_1) L_0^j + 2\delta \int_0^\infty \text{Std}(L_0^j, k; \text{Call}) dk \right).$$

3.2. Cap/floor in arrears. For the cap, $p(x) = (x - K)^+$ and we apply the above proposition with $f(x) = x(x - K)^+$. The expectation is then

$$\mathbb{E}^{t_1} \left[L_\theta^j p(L_\theta^j) \right] = K \text{Base}(L_0^j, K; \text{Call}) + 2 \int_K^\infty \text{Base}(L_0^j, k; \text{Call}) dk.$$

For the floor, $p(x) = (K - x)^+$ and we apply the above equivalent of the proposition with $f(x) = x(K - x)^+$. The expectation is then

$$\mathbb{E}^{t_1} \left[L_\theta^j p(L_\theta^j) \right] = K \text{Base}(L_0^j, K; \text{Put}) + 2 \int_0^K \text{Base}(L_0^j, k; \text{Put}) dk.$$

The price is

$$V^{\text{IA}} = \frac{1}{\beta} \left((1 + \delta K) \text{Std}(L_0^j, K; T) + 2\delta \int_a^b \text{Std}(L_0^j, k; T) dk \right)$$

with $[a, b] = [K, +\infty]$ for caps and $[a, b] = [0, K]$ for floors.

Remark: The swap in arrears price is equal to the price of the cap with strike 0.

4. IBOR-WITH-DELAY

Let θ be the fixing date of a Ibor rate for the period $[t_0, t_1]$. The payment of an Ibor-related product (swap floating coupon or cap/floorlet) take place on $t_p > \theta$ and its pay-off is $p(L_\theta^j)$. For floating coupons, $p(x) = x$ and for caps, $p(x) = (x - K)^x$ with K the strike.

In the $P^D(\cdot, t_1)$ numeraire, the price of the product paid on t_p can be written as

$$V^{\text{Delay}} = P^D(0, t_1) \mathbb{E}^{t_1} \left[\frac{P^D(\theta, t_p)}{P^D(\theta, t_1)} p(L_\theta^j) \right].$$

Let η be the ratio of difference between the time, i.e.

$$\eta = \frac{t_p - t_0}{t_1 - t_0}.$$

We approximate the ratio of discount factors required in the above expectation by

$$\frac{P^D(\theta, t_p)}{P^D(\theta, t_1)} = \frac{P^D(0, t_p)}{P^D(0, t_1)} \frac{1}{(1 + \delta L_0^j)^{1-\eta}} (1 + \delta L_\theta^j)^{1-\eta}.$$

The approximation is based on the idea that any rate dynamic can be represented by the compounded Ibor rate.

To compute the price of the delay product one only need compute

$$\mathbb{E}^{t_1} \left[(1 - \delta L_\theta^j)^{1-\eta} p(L_\theta^j) \right].$$

Similarly, this can be computed with the replication argument and $f(x) = (1 + \delta x)^{1-\eta} p(x)$.

4.1. Cap. In the cap case, the expectation is

$$f'(K) \text{Base}(L_0^j, K; \text{Call}) + \int_K^\infty f''(k) \text{Base}(L_0^j, k; \text{Call}) dk.$$

4.2. **Floor.** In the floor case, the expectation is

$$f'(K) \text{Base}(L_0^j, K; \text{Call}) + \int_0^K f''(k) \text{Base}(L_0^j, k; \text{Put}) dk.$$

The price is

$$V^{\text{Delay}} = P^D(0, t_1) \frac{P^D(0, t_p)}{P^D(0, t_1)} \frac{1}{(1 + \delta L_0^j)^{1-\eta}} \left(\omega(1 + \delta K)^{1-\eta} \text{Base}(L_0^j, K; T) + \int_a^b f''(k) \text{Base}(L_0^j, k; T) dk \right)$$

with $[a, b] = [K, +\infty]$ for caps and $[a, b] = [0, K]$ for floors.

The function derivatives are (where non-zero)

$$\begin{aligned} f(x) &= \omega(1 + \delta x)^{1-\eta}(x - K) \\ f'(x) &= \omega(1 + \delta x)^{-\eta}((1 - (1 - \eta)\delta K) + (2 - \eta)\delta x) \\ f''(x) &= \omega(1 + \delta x)^{-\eta-1}(1 - \eta)\delta(2 + \eta\delta K + (2 - \eta)\delta x). \end{aligned}$$

Remark: When the fixing is in arrears, $\eta = 0$ and $f'(x) = 1 - \delta K + 2\delta x$ and $f''(x) = 2\delta$. The Ibor-with-delay price reduces to the Ibor-in-arrears price.

5. IMPLEMENTATION

The instruments are described in the classes `CouponIbor` and `CapFloorIbor`. The instrument descriptions are flexible enough to cover standard, in-arrears and with-delay instruments in the same object.

The in-arrears pricing method is implemented in the class `CapFloorIborInArrearsReplicationMethod`. It works with any cap/floor base pricing method. Currently the methods implemented for caps/floors are `CapFloorIborSABRMethod` and `CapFloorIborSABRExtrapolationRightMethod`.

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