Equity Variance Swap with Dividends

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Abstract

We present a discussion paper on how to price equity variance swaps in the presence of known (cash and proportional) dividends.

1 Variance swaps on non-dividend paying stock

In this section we present the standard theory of pricing a variance swap. Good references for this are Demeterfi(1999), Neuberger(1992) and Carr(1998).

The payoff of a standard variance swap[1] is given by:

$$VS(T) = N_{var}\left[\frac{A}{n}\sum_{i=1}^{n}\left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2 - K^2\right] \tag{1}$$

where S_i is the closing price of a stock or index on the i^{th} observation date, the annualisation factor A is set by the contract (and usually 252), N_{var} is the variance notional and K^2 (the variance strike) is usually chosen to make the initial value zero. The notional is often quoted in units of volatility (i.e. \$1M per vol), N_{vol} with $N_{var} = \frac{N_{vol}}{2K}$.

Since all the other terms are fixed by the contract, we will focus on the realised variance part

$$RV(T) = \sum_{i=1}^{n}\left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2 \tag{2}$$

1.1 The Black-Scholes World

If the stock follows the process

$$\frac{dS_t}{S_t} = rdt + \sigma dW \tag{3}$$

then the returns are given by

$$R_i \equiv \log\left(\frac{S_i}{S_{i-1}}\right) = (r - \sigma^2/2)\Delta t + \sigma \Delta W \tag{4}$$

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[1]The name *swap* is a bit of a misnomer, as it is a forward contract on the realised variance

That is, they are normally distributed with mean $(r - \sigma^2/2)\Delta t$ and variance $\sigma^2 \Delta t$ Ignoring the drift term (which will be negligible next to the diffusion term for a period of one day), we can write the realised variance as

$$RV(T) = \sigma^2 \Delta t \sum_{i=1}^{n} Z_i^2 \tag{5}$$

where Z_i are standard normal random variables. Hence the realised variance has a chi-squared distribution with n degrees of freedom scaled by $\sigma^2 \Delta t$ - which is a Gamma distribution $\Gamma(n/2, 2\sigma^2 \Delta t)$. The expected variance is simply

$$\mathbb{E}[RV(T)] = \sigma^2 T \tag{6}$$

but even in the Black-Scholes world, the realised variance can vary greatly from this.[2]

1.2 Other diffusions

By considering the Taylor series of $\log\left(\frac{S_i}{S_{i-1}}\right)$ and $\left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2$ about S_{i-1}, it is easy to show that

$$\left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2 \approx -2\log\left(\frac{S_i}{S_{i-1}}\right) + 2\frac{S_i - S_{i-1}}{S_{i-1}} + \mathcal{O}\left(\frac{(S_i - S_{i-1})^3}{S_{i-1}^3}\right) \tag{7}$$

Summing the terms gives Neuberger's formula, Neuberger(1992).

$$RV(T) \approx -2\log\left(\frac{S_T}{S_0}\right) + 2\sum_{i=1}^{n} \frac{1}{S_{i-1}}(S_i - S_{i-1}) \tag{8}$$

What this says is that you can replicate the realised variance up to the expiry, T, by holding $2\log(S_0)$ zero coupon bonds with expiry at T, shorting two log-contracts[3] and daily delta hedging with a portfolio who's delta is $2/S_{i-1}$.

The expected value of the realised variance (or just expected variance) is

$$EV(T) = 2\log(F_T) - 2\mathbb{E}[\log(S_T)] \tag{9}$$

and therefore the fair value of the variance strike is

$$K = \sqrt{\frac{2A}{n}\left(\log(F_T) - \mathbb{E}[\log(S_T)]\right)} \tag{10}$$

where F_T is the forward price. This is model independent provided the stock process is a diffusion, Gatheral(2006).

1.3 Static Replication

Any twice differentiable function, $H(x)$ can be written exactly as (Carr(1998))

$$H(x) = H(x_0) + (x - x_0)H'(x_0) + \int_0^{x_0} H''(z)(z-x)^+ dz + \int_{x_0}^{\infty} H''(z)(x-z)^+ dz \tag{11}$$

[2] The fractional error is approximately $\sqrt{2/n}$

[3] A log-contact has the payoff $\log(S_T)$ at expiry - these are not liquid but can be statically replicated with strips of Europeans puts and calls

for $x \geq 0$. The (non-discounted) value of a derivative at time t with a payoff $H(S_T)$ at T is $\mathbb{E}[H(S_T)|\mathcal{F}_t]$. Expanding $H(S_T)$ using equation 11 we find

$$\begin{aligned}\mathbb{E}[H(S_T)|\mathcal{F}_t] &= H(s^*) + (\mathbb{E}[S_T] - s^*)H'(s^*) \\ &\quad + \int_0^{s^*} H''(k)\mathbb{E}[(k - S_T)^+]dk + \int_{s^*}^{\infty} H''(k)\mathbb{E}[(S_T - k)^+]dk\end{aligned} \tag{12}$$

Choosing $s^* = F_T$ and recognising that $\mathbb{E}[(S_T - k)^+]$ and $\mathbb{E}[(k - S_T)^+]$ are (non-discounted) call and put prices at strike k, we can write

$$\mathbb{E}[H(S_T)|\mathcal{F}_t] = H(F_T) + \int_0^{F_T} H''(k)P(k)dk + \int_{F_T}^{\infty} H''(k)C(k)dk \tag{13}$$

So in principle, one can replicate any payoff that is a function of the underlying at expiry, by an infinite strip of puts and calls on the underlying, with weights $H''(k)$. For our log contract we have $H''(k) = -1/k^2$, so the expected variance can be written as

$$EV(T) = 2\left(\int_0^{F_T} \frac{P(k)}{k^2}dk + \int_{F_T}^{\infty} \frac{C(k)}{k^2}dk\right) \tag{14}$$

In practise only a finite number of liquid call and put prices are available in the market and the expiries may not match that of the variance swap. What is needed is an arbitrage free interpolation of option prices. For a call with expiry T and strike k we have

- A call with a zero strike equals the forward $C(T, 0) = F_T$
- Positive calendar spreads, $\frac{\partial C(T,k)}{\partial T} \geq 0$
- Call price is monotonically deceasing in strike, $\frac{\partial C(T,k)}{\partial k} \leq 0$
- Positive butterflies, $\frac{\partial^2 C(T,k)}{\partial k^2} \geq 0$
- High strike limit, $c(T, \infty) = 0$
- If the underlying cannot reach zero (e.g. Exponential Brownian Motion (EBM)) then, $\frac{\partial C(T,0)}{\partial k} = -1$, otherwise (e.g. SABR) $\frac{\partial C(T,0)}{\partial k} > -1$

The corresponding conditions for puts follow from put-call parity. We carry out the interpolation in implied volatility space, using arbitrage free smile models for interpolation in the strike direction - this is detailed in White(2012). To extrapolate to strikes outside the range of market quotes, we use a shifted log-normal model. This is just the Black option pricing formula where the forward and the volatility are treated as free parameters which are calibrated so that either the last two option prices are matched, or the price and dual delta[4] of the last option is matched. This is done separately for low strikes (puts) and high strikes (calls).

The last practical detail is to find an upper cut-off for the integral. Using the monotonically decresing property, we know that for a cut-off of A we have

$$\int_A^{\infty} \frac{C(k)}{k^2}dk \leq \int_A^{\infty} \frac{C(A)}{k^2}dk = \frac{C(A)}{A} \tag{15}$$

So having integrated up to A we have an upper limit on the error.

[4]dual delta is the rate of change of a option price with respect to the strike

Using L'Hôpital's rule, we find

$$\lim_{k\to 0} \frac{P(k)}{k^2} = \frac{1}{2}\frac{\partial^2 P(T,0)}{\partial k^2} \tag{16}$$

Which is half the probability density at zero, and is zero for our shifted log-normal extrapolation. This limit on the integrand should be used in any numerical integration routine. Alternatively we have for very small A

$$\int_0^A \frac{P(k)}{k^2} dk \approx \frac{P(A)}{2A} + \theta \tag{17}$$

where θ is the probability mass at zero (zero for exponential Brownian motion but non-zero for SABR etc). This can again be used to set a limit on the error.

2 Dividends

In this section we follow the approach of Buehler (2010) and Bermúdez (2006)[5].

Companies make regular dividend payments (e.g. every six months) of a fixed amount per share, with the amount announced well in advance[6], so dividends over a 1-2 year horizon can be treated as fixed cash payments (per share) regardless of the share price. Further in the future, the dividend payment is likely to depend on company performance, hence we can model payments as being a fixed proportion of the share price on the ex-dividend date. In general dividends are paid[7] at times $\tau_1, \tau_2, ...$ from today, with a cash amount α_i and a proportional about β_i. If the stock price immediately before the dividend payment is $S_{\tau_i -}$, then the price just after must be

$$S_{\tau_i} = S_{\tau_i -}(1-\beta_i) - \alpha_i \tag{18}$$

If the stock price follows a diffusion process, then this implies that it can go negative. One way to avoid this is by making the dividend payment a (non-affine) function of the stock price, $D_i(S_{\tau_i -})$ such that $D_i(S_{\tau_i -}) \leq S_{\tau_i -}$ (see Haug(2003) and Nieuwenhuis(2006)). Bermúdez (2006) and Buehler (2010) show that their affine dividends assumption leads to a modified stock price process[8].

Consider a forward contract to deliver one share at maturity T in exchange for a payment of k. If we start with δ_t shares then we will receive a dividend of $\delta_t(S_{\tau_1 -}\beta_1 + \alpha_1)$ on the first dividend date after t. Rewriting this in terms of the post-dividend price S_{τ_1} we have

$$\text{Dividend Payment}_1 = \delta_t \left(\frac{\beta_1}{1-\beta_1} S_{\tau_1} + \frac{\alpha_1}{1-\beta_1} \right) \tag{19}$$

We use the part that is proportional to S_{τ_1} to buy new stock immediately after the payment, then the stock holding will be

$$\delta_{\tau_1} = \frac{\delta_t}{1-\beta_1} \tag{20}$$

Our residual cash may be written as $\delta_{\tau_1}\alpha_1$.

If we repeat the procedure for each dividend payment, then at T we will own $\delta_T = \delta_t / \prod_{j:t<\tau_j \leq T}(1-\beta_j)$ shares. To have exactly one share at maturity, set

$$\delta_t = \prod_{j:t<\tau_j \leq T} (1-\beta_j) \tag{21}$$

[5]It should be noted that both the book and the paper contain considerable typos in the mathematics
[6]Although there may be no legal obligation to actually pay this amount if the company's performance is poor.
[7]We assume the ex-dividend date and the payment date coincide.
[8]They also consider credit risk and repo rates, which we ignore are present.

Then at some time s with $t < s \leq T$ we own

$$\delta_s = \prod_{j:s<\tau_j\leq T} (1-\beta_j) \tag{22}$$

shares. At each dividend date we also have residual cash equal to $\delta_{\tau_i}\alpha_i$. As these are fixed, we can realise their value now by selling $\delta_{\tau_i}\alpha_i$ zero coupon bonds expiring at τ_i. The total value of the sale is

$$\sum_{j:t<\tau_j\leq T} \delta_{\tau_i}\alpha_j P(t,\tau_j) \tag{23}$$

If we define the growth factor as

$$R(t,T) = \frac{1}{P(t,T)} \prod_{j:t<\tau_j\leq T} (1-\beta_j) \tag{24}$$

then the total initial investment that guarantees one share at time T is

$$P(t,T)R(t,T)\left(S_t - \sum_{j:t<\tau_j\leq T} \frac{\alpha_j}{R(t,\tau_j)}\right) \tag{25}$$

We can fund this by selling k zero coupon bonds with maturity T. The value of k that makes the initial investment zero is the forward price, therefore

$$F(t,T) = R(t,T)\left(S_t - \sum_{j:t<\tau_j\leq T} \frac{\alpha_j}{R(t,\tau_j)}\right) \tag{26}$$

Since the share price cannot become negative, neither can the forward. This implies that $S_t \geq \sum_{j:t<\tau_j\leq T} \frac{\alpha_j}{R(t,\tau_j)}$, and since this must be independent of the arbitrary maturity, T, of a forward, we must have

$$S_t \geq D_t \equiv \sum_{j:\tau_j>t} \frac{\alpha_j}{R(t,\tau_j)} \tag{27}$$

That is, the dividend assumption imposes a floor on the stock price equal to the growth factor discounted value of all future cash dividends.[9]

As a result of the floor on the stock price, the Black model, $S_t = F_t X_t$,[10] where X_t is a EBM with $\mathbb{E}[X_t] = 1$, is no longer valid since $\mathbb{P}(S_t < D_t) > 0$.[11]

Bermúdez (2006) and Buehler (2010) propose

$$S_t = (F_t - D_t)X_t + D_t \tag{28}$$

with $\mathbb{E}[X_t] = 1$ and $X_t \geq 0$.[12] The stock price experiences randomness just on the portion above the floor D_t. X_t is known as the *pure* stock process.

[9] We have overloaded the letter D - D_t is the growth factor discounted value of all future cash dividends, while $D(\tau)$ is the dividend paid at time τ

[10] $F_t \equiv F(0,t)$

[11] Any other process for X_t is also invalid for the same reason.

[12] Their model also includes credit risk, so they impose $X_t > 0$ and have the stock price and all future dividends fall to zero on a default. The setup with no credit risk and $X_t \geq 0$ implied that the stock becomes a bond with fixed coupons if x_t becomes zero.

2.1 European Options

Consider the price at time t of two european call options, one with expiry τ^- (i.e. just before the dividend payment) and one with expiry τ (i.e. just after the payment). We write the prices as $C_t(\tau^-, k^-)$ & $C_t(\tau, k)$. The first payoff is

$$(S_{\tau^-} - k^-)^+ = \left(\frac{S_\tau + \alpha}{1-\beta} - k^-\right)^+ = \frac{1}{1-\beta}\left(S_\tau - [(1-\beta)k^- - \alpha]\right)^+ \tag{29}$$

If we let $k = (1-\beta)k^- - \alpha$ then the payoff of the first option is just $1/(1-\beta)$ times that of the second. It then follows that

$$(1-\beta)C_t(\tau^-, k^-) = C_t(\tau, (1-\beta)k^- - \alpha) \tag{30}$$

If the options are priced using Black's formula with the same volatility, then the condition of equation 30 does not hold. Conversely, if the condition does hold, and (Black) implied volatilities are found, they will jump up across the dividend date. Once again, the Black model, or indeed any model that has a continuous implied volatility surface, is not consistent with the dividend assumption.

Another consideration is what happens to the price of an option with expiry, T, as the current time, t, moves over a dividend date. We now write the call price as $C(t, S_t, T, k)$. Clearly

$$C(\tau^-, S_{\tau^-}, T, k) = C(\tau, S_\tau, T, k) \tag{31}$$

This is true for any European option (e.g. a log-payoff) with $T > \tau$. This condition must be applied when solving a backwards PDE for option prices, as discussed later.

2.2 Options on the Pure Stock

Define a call option on the pure stock as

$$\tilde{C}(T, x) = \mathbb{E}[(X_T - x)^+] \tag{32}$$

This is related to a real call by

$$\begin{aligned} \tilde{C}(T, x) &= \frac{1}{P(0,T)(F_T - D_T)} C\left(T, (F_T - D_T)x + D_T\right) \\ C(T, k) &= P(0,T)(F_T - D_T)\tilde{C}\left(T, \frac{k - D_T}{F_T - D_T}\right) \end{aligned} \tag{33}$$

Hence if the dividend structure is known, market prices for calls (and puts) can be converted into pure call prices - which is turn can be converted to a pure implied volatility by inverting the Black formula in the usual way.

Figure 1 shows the implied volatility surface if the underlying prices are generated from a flat pure implied volatility surface at 40%, and the pure implied volatility surface if the prices are generated from a flat implied volatility surface at 40%. These shapes are observed in Buehler(2010), but without the jumps at the dividend dates. The condition leading to equation 30 means we expect a jump in implied volatility if the pure implied volatility is flat and vice versa. As Buehler does not state the dividends schedule, we cannot make a direct comparison. However if we increase the dividend frequency ten-fold and decrease each amount ten-fold (which may be more the case for indices)[13], then the implied volatility surface (for a flat pure implied volatility surface) becomes as shown in figure 2.

[13] We have cash only dividends in year one, proportional only after 3 years and a linear switch over in year 2

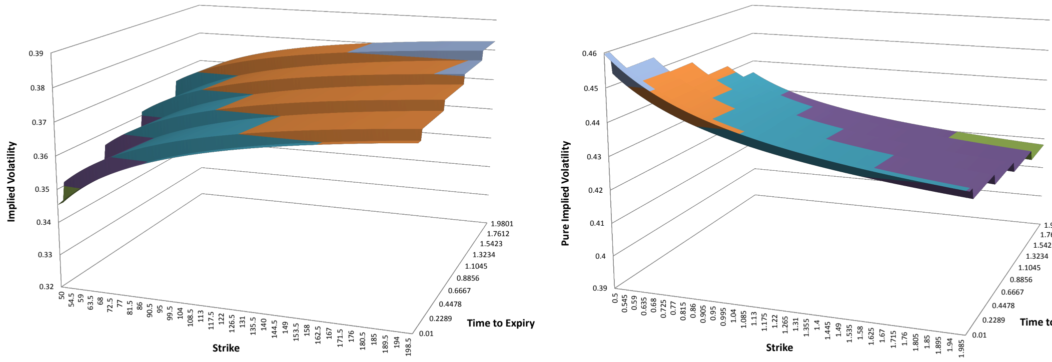

Figure 1: Implied Volatility (left) and Pure Implied volatility (right) for an initial stock price is 100, dividends every six months and the next is in one month, with a proportional part equal to 1% of the stock price ($\beta = 0.01$) and a cash part of 1.0, and the risk free rate is 5%. In each case the prices were generated from a flat pure implied volatility surface (left), or a flat implied volatility surface (right) - both at 40%

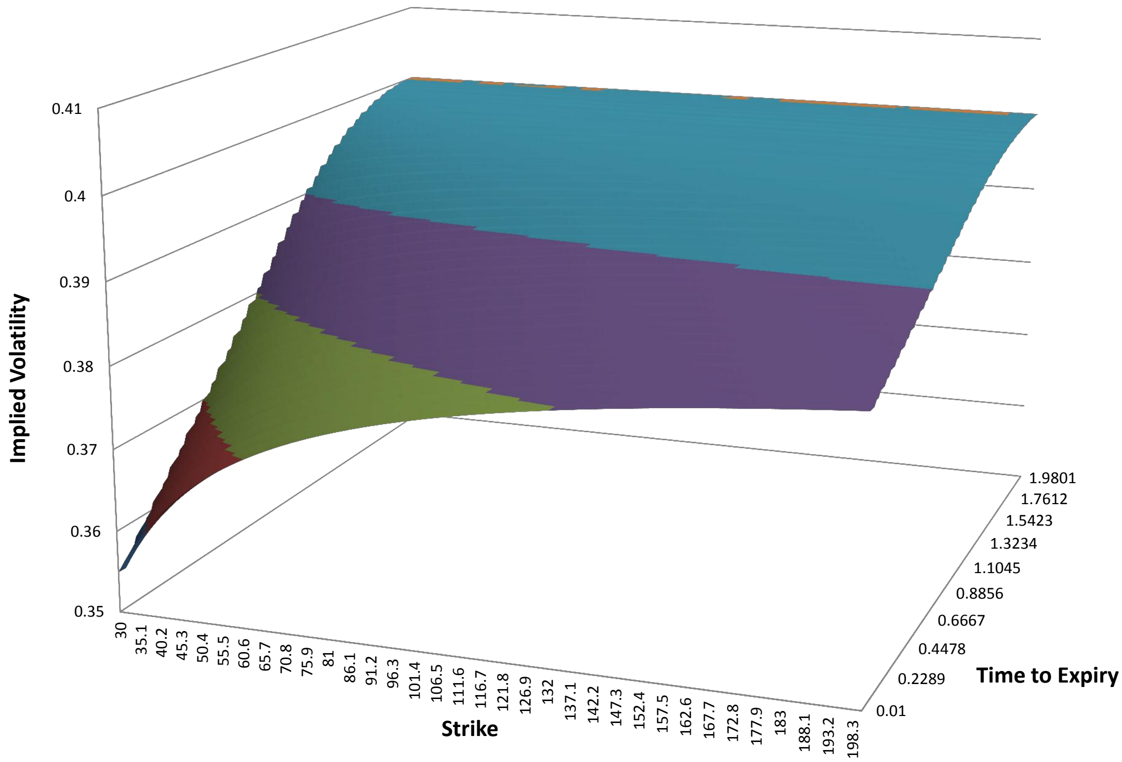

Figure 2: Implied Volatility surface for a pure implied volatility surface of 40%, and 20 dividend payments per year

2.3 Dupire Local Volatility for Pure Stock Process

Following exactly the same argument as Dupire(1994), if a unique solution to the SDE

$$\frac{dX_t}{X_t} = \sigma^X(t, X_t)dW_t \tag{34}$$

exists, then the pure stock local volatility is given by:

$$\sigma^X(t, X_t)^2 = 2\frac{\frac{\partial \tilde{C}}{\partial T}}{x^2 \frac{\partial^2 \tilde{C}}{\partial x^2}} \tag{35}$$

Once we have the pure option prices (or equivalently the pure implied volatilities) from market prices, we can proceed as in White(2012) to construct a smooth Pure Local Volatility surface, and numerically solve PDEs without having to impose jump conditions at the dividend dates. Conversely, if we solve a PDE expressed in the stock price, then at dividend dates the nodes must be shifted such that for the i^{th} stock price node $s^i_\tau = s^i_{\tau^-} - D(S_{\tau^-})$, where $D(S_{\tau^-})$ is the dividend amount, and the option price maps according to equation 31. Between dividend dates the local volatility is

$$\sigma(t, S_t) = \mathbf{1}_{S_t > D_t} \frac{S_t - D_t}{S_t} \sigma^X\left(t, \frac{S_t - D_t}{F_t - D_t}\right) \tag{36}$$

3 Expected Variance in the Presence of Dividends

3.1 Proportional Dividends Only

The stock price process can be written as

$$\frac{dS_t}{S_{t^-}} = r_t dt + \sigma_t(S_{t^-})dW_t - \sum_j \beta_j \delta(\tau_j) \tag{37}$$

where the volatility is a function of the stock price immediately prior to any jump (and possibly another stochastic factor), and $\delta(\tau_j)$ indicates that jumps only occur at one of the dividend dates τ_j.

If we define $y_t = \log(S_t)$ then

$$dy_t = (r_t - \sigma_t^2/2)dt + \sigma_t dW_t + \sum_j \log(1 - \beta_j)\delta(\tau_j) \tag{38}$$

and

$$y_s = y_t + \int_t^S (r_{t'} - \sigma_{t'}^2/2)dt + \int_t^S \sigma_{t'} dW_{t'} + \sum_{j: t < \tau_j \le s} \log(1 - \beta_j) \tag{39}$$

Which is a formal solution since in general σ_t is a function of y_t. The expected squared return is

$$\mathbb{E}\left[\ln^2\left(\frac{S_i}{S_{i-1}}\right)\right] \approx \int_{t_{i-1}}^{t_i} \sigma_t^2 dt + \sum_j \log^2(1 - \beta_j)\delta_{\tau_j = t_i} \tag{40}$$

which is the quadratic variance of the continuous part of the log-process, plus the contribution (if any) from the dividend. The total expected variance is then

$$EV(T) \approx \int_0^T \sigma_t^2 dt + \sum_{j: \tau_j \le T} \log^2(1 - \beta_j) \tag{41}$$

Of course, if dividends are corrected for (as is usual for single stock), then the second term is absent.

Again from equation 45 the expected value of a payoff of $\log(S_T)$ is

$$\begin{aligned}\mathbb{E}[\log(S_T)] \equiv \mathbb{E}[y_T] &= \log(S_0) + \int_0^T r_t dt + \sum_{j:\tau_j \leq T} \log(1-\beta_j) - \frac{1}{2}\int_0^T \sigma_t^2 dt \\ &= \log(F_T) - \frac{1}{2}\int_0^T \sigma_t^2 dt\end{aligned} \tag{42}$$

where the replacement of the first three terms by the forward comes from equation 26. It then follows immediately that

$$EV(T) = -2\mathbb{E}\left[\log\frac{S_T}{F_T}\right] + \sum_{j:\tau_j \leq T} \log^2(1-\beta_j) \tag{43}$$

So for proportional dividends only, where an adjustment is made for the dividend, the expected variance is the same as the no dividend case,[14] and otherwise there is a trivial adjustment.

3.2 With Cash Dividends

Following the argument in Buehler(2006), the realised variance can be expressed as

$$RV(T) = \sum_{i=1}^{n} \log^2\left(\frac{S_{i-}}{S_{i-1}}\right) + \sum_{j:t<\tau_j \leq T} \log^2\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right) \tag{44}$$

S_{i-} is the stock price just before the dividend payment (if one occurs on that observation date), hence the first term is the realised variance of the continuous part, $RV(T)_{\text{cont}}$. If dividends are corrected for, then the second term disappears, and the realised variance is the same as the realised variance of the continuous part, as above.

The SDE for the log of the stock price is given by

$$d(\log S_t) = \frac{1}{S_{t^-}} dS_t - \frac{1}{2} dRV_{\text{cont}} + \sum_j \log\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right)\delta_{\tau_j}(dt) \tag{45}$$

The last term simply means that at dividend dates the log of the stock jumps by $\log\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right)$, but is otherwise continuous. Integrating equation 45 and combining with equation 44 gives

$$\begin{aligned}RV(T) =& -2\ln\frac{S_T}{S_0} + 2\int_0^T \frac{1}{S_{t^-}} dS_t \\ &+ 2\sum_{j:\tau_j \leq T} \log\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right) + \sum_{j:t<\tau_j \leq T} \log^2\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right) \\ =& -2\ln\frac{S_T}{S_0} + 2\int_0^T r_t dt + 2\int_0^T \sigma_t dW_t \\ &+ 2\sum_{j:\tau_j \leq T} \log\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right) + \sum_{j:t<\tau_j \leq T} \log^2\left(\frac{S_{\tau_j}(1-\beta_j)}{S_{\tau_j}+\alpha_j}\right)\end{aligned} \tag{46}$$

[14]The big caveat here is that option prices will not be the same with and without dividends, so the value will be different.

Taking expectations we arrive at the result

$$\begin{aligned} EV(T) &= -2\mathbb{E}\left[\log\frac{S_T}{S_0}\right] - 2\log(P(0,T)) + 2\sum_{j:\tau_j\le T}\mathbb{E}\left[G_j(S_{\tau_j})\right] \\ \text{where}\quad G_j(s) &\equiv \log\left(\frac{s(1-\beta_i)}{s+\alpha_i}\right) + \frac{1}{2}\log^2\left(\frac{s(1-\beta_i)}{s+\alpha_i}\right) \end{aligned} \tag{47}$$

Unlike the pure proportion case, we cannot rewrite this in terms of the forward as

$$\log(F_T) \neq \log(S_0) - \log(P(0,T)) + \sum_{j:\tau_j\le T}\mathbb{E}\left[\log\left(\frac{S_{\tau_j}(1-\beta_i)}{S_{\tau_j}+\alpha_i}\right)\right] \tag{48}$$

The correction term is a (twice differentiable) function of the stock price just after a dividend payment, so can be priced with a strip of call and puts with expiry τ_j as in equation 13. When dividends are corrected for (the usual case for single stock), then last term is removed. An alternative derivation of this result is given in appendix A.

We now have a mechanical procedure to value variance swaps when dividends are paid on the underlying stock and the stock price is a pure diffusion between dividend dates.

- Estimate the future dividend schedule. This can come for declared dividends (for short time periods), equity dividend futures and calendar spreads.
- Collect prices of all European options on the underlying stock or index[15], and convert to prices of pure put and call options
- Construct the pure local volatility surface
- solve the forward PDE up to the expiry of the variance swap
- Convert the pure option prices on the PDE grid, to pure implied volatility to form a continuous (interpolated) surface
- Convert back to option prices at arbitrary expiry and strike
- Compute the value of the log-contact and the correction at each dividend date using equation **??**.

3.3 The Buehler Paper

In Buehler (2010), the realised variance which is not adjusted for dividends is given as

$$\begin{aligned} RV(T) &= -2\log\frac{S_T}{S_0} + 2\sum_{i=1}^{n}\left(\frac{S_{t_i}}{S_{t_{i-1}}}-1\right) + 2\sum_{j:\tau_j\le T}E_j(S_{\tau_j}) \\ \text{where}\quad E_j(s) &= \log\frac{s(1-\beta_j)}{s+\alpha_j} + \frac{s\beta_j+\alpha_j}{s+\alpha_j} - \left(\frac{s\beta_j+\alpha_j}{s+\alpha_j}\right)^2 \end{aligned} \tag{49}$$

[15] Most equity options are American style, but we will ignore this important detail for now.

Taking expectations, we find the expected variance is

$$
\begin{aligned}
EV(T) &= -2\mathbb{E}\left[\log \frac{S_T}{S_0}\right] - 2\log(P(0,T)) + 2\sum_{i=1}^{n} \mathbb{E}\left[\hat{E}_i(S_{\tau_i})\right] \\
\text{where} \quad \hat{E}_j(s) &= \log \frac{s(1-\beta_j)}{s+\alpha_j} - \left(\frac{s\beta_j + \alpha_j}{s+\alpha_j}\right)^2
\end{aligned}
\tag{50}
$$

This is not the same as our equation 47, since the last term (which only applies if dividends are not corrected for) is $-\left(\frac{s\beta_j+\alpha_j}{s+\alpha_j}\right)^2$ rather than $\frac{1}{2}\log^2\left(\frac{s(1-\beta_i)}{s+\alpha_i}\right)$. We believe this is a mistake in the paper, Buehler (2010), and have confirmed this using Monte Carlo methods described in the next section.

4 Numerical Testing

We present four test cases with exaggerated parameters to highlight any errors. All cases have a spot of 100, a pure local volatility of 30%, a fixed interest rate of 10%, a single dividend payment at 0.85 years and expiry of the variance swap at 1.5 years. The first two cases have $\alpha = 0$ and $\beta = 0.4$, the former with and the latter without dividends corrected for in the realised variance. The second two cases have $\alpha = 30$ and $\beta = 0.2$, again with and without dividends correction.

Since the pure local volatility is flat, we can find call and put prices from the modified Black formula of equation 33, which allows us to find the expected variance using equations 47 with the static replication of equation 13 (once at expiry and once at the dividend date). In the case of pure proportion dividends (first two cases), we have a simple analytical answer via equation 43[16] . All the numerical methods do not depend on a flat pure local volatility surface.

Another method is to solve the backwards PDE with the log-payoff as the initial condition. This can be done using the pure stock or the actual stock as the spacial variable - in the latter case the jump condition must be applied at the dividend. Since this method does not calculate the terms $G(S)$, it is only applicable to the first test case ($\alpha = 0$ with dividends corrected for in the realised variance).

Solving the forward PDE for the pure call prices up to expiry, gives actual call and put prices on a expiry-strike grid, which can be used as in the static replication case to price all four cases. This in our principle numerical method.

Finally, it is simple to set up Monte Carlo to calculate expected variance in all four cases.[17]

The results are shown in table 1 below, expressed in terms of K where K^2 is the variance strike (i.e. $K = \sqrt{EV(T)/T}$). The agreement to 4 dp of the forward PDE to the Monte Carlo, gives use confidence in the method.

5 Concussion

We have reviewed the pricing of variance swaps, and the affine dividend model of Buehler *et al.* We believe that there is a mistake in the paper Buehler(2010), and that the expression presented here is correct.

A more general dividend model would require more numerical work to calibrate and price variance swaps.

[16] For a flat pure local volatility, σ^x we have $\mathbb{E}\left[\log \frac{S_T}{F_T}\right] = -\frac{1}{2}(\sigma^x)^2$

[17] Since the purpose of the Monte Carlo is to test the other methods, we are free to use a very large number of paths.

	Case1	Case2	Case3	Case4
Analytic	0.3	0.5138	N/A	N/A
Static Replication	0.3000	0.5138	0.2463	0.6035
Backwards PDE (pure stock)	0.3000	N/A	N/A	N/A
Backwards PDE (actual stock)	0.3000	N/A	N/A	N/A
Forward PDE	0.3000	0.5138	0.2463	0.6035
Monte Carlo	0.3000	0.5136	0.2461	0.6035

Table 1: The expected variance expressed as $K = \sqrt{EV(T)/T}$ for the 4 test cases and 5 numerical methods discribled above.

A Alternative Derivation

Assuming that S_i is the closing price on date t_i and that dividends are paid just before this, so that

$$S_i = S_{i^-} - D_i \tag{51}$$

where D_i is the amount of the dividend (which is zero except for dividend dates). The realised variance can be written as

$$\begin{aligned} RV(T) &= \sum_{i=1}^{n} \log^2\left(\frac{S_i}{S_{i-1}}\right) = \sum_{i=1}^{n} \log^2\left(\frac{S_{i^-} - D_i}{S_{i^-}} \frac{S_{i^-}}{S_{i-1}}\right) \\ &= \sum_{i=1}^{n} \log^2\left(\frac{S_{i^-}}{S_{i-1}}\right) + \sum_{j:t<\tau_j\leq T} \log^2\left(\frac{S_{\tau_j}}{S_{\tau_j} + D_j}\right) \\ &\quad + 2\sum_{j:t<\tau_j\leq T} \log\left(\frac{S_{\tau_j}}{S_{\tau_j} + D_j}\right) \log\left(\frac{S_{\tau_j^-}}{S_{\tau_j^- - \Delta t}}\right) \end{aligned} \tag{52}$$

The last term (which is the return of continuous part multiplied by the return due to the dividend payment) is negligible and can be ignored. This is the same as the result show in equation 44. The first term (which is the squared returns of the continuous part), can be approximated as in equation 7.

$$\log^2\left(\frac{S_{i^-}}{S_{i-1}}\right) \approx -2\log\left(\frac{S_{i^-}}{S_{i-1}}\right) + 2\frac{S_{i^-} - S_{i-1}}{S_{i-1}} \tag{53}$$

We rewrite the first term as

$$\log\left(\frac{S_{i^-}}{S_{i-1}}\right) = \log\left(\frac{S_i}{S_{i-1}}\right) + \log\left(\frac{S_i + D_i}{S_i}\right) \tag{54}$$

Hence

$$\begin{aligned} RV(T) \approx{}& -2\log(S_T/S_0) + 2\sum_{i=1}^{n} \frac{S_{i^-} - S_{i-1}}{S_{i-1}} \\ &+ 2\sum_{j:t<\tau_j\leq T} \log\left(\frac{S_{\tau_j}}{S_{\tau_j} + D_j}\right) + \sum_{j:t<\tau_j\leq T} \log^2\left(\frac{S_{\tau_j}}{S_{\tau_j} + D_j}\right) \end{aligned} \tag{55}$$

If we take the expectation and set $D_j = \frac{\beta_j S_{\tau_j} + \alpha_j}{1-\beta_j}$ we arrive at the result in equation 47.

References

Bermúdez, Ana (2006). Equity hybrid derivatives. Wiley Finance.

Buehler, Hans (2006). Options on Variance: Pricing and Hedging. IQPC Volatility Trading Conference. http://www.quantitative-research.de/dl/IQPC2006-2.pdf

Buehler, Hans (2010). Volatility and Dividends. Working paper available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1141877]

Carr, Madan (1998). Towards a Theory of Volatility Trading. In "Volatility: New Estimation Techniques for Pricing Derivatives," R. Jarrow (ed.) RISK Publications, London.

Demeterfi, Derman, Kamal, Zou (1999). More Than You Ever Wanted To Know About Volatility Swaps. Goldman Sachs Quantitative Strategies Research Notes.

Dupire, B. (January 1994). Pricing with a Smile. Risk Magazine, Incisive Media.

Gatheral, Jim. (2006). The Volatilty Surface. Wiley Finance.

Haug, Espen; Haug, Jorgen; Lewis, Alan (2003). Back to Basics: a new approach to the discrete dividend problem.

Neuberger, A (1992). Volatility Trading. London Business School working paper.

Nieuwenhuis, J.W.; Vellekoop, M. H. (2006). Efficient Pricing of Derivatives on Assets with Discreate Dividends. Applied Mathematical Finance.

White, Richard R (2012). Local Volatility. OpenGamma working paper.http://docs.opengamma.com/display/DOC/Quantitative+Papers+and+Working+Notes