BINORMAL WITH CORRELATION BY STRIKE APPROACH TO CMS SPREAD PRICING

OPENGAMMA QUANTITATIVE RESEARCH

Abstract. The pricing of CMS spread using a binormal with correlation by strike approach is described.

1. Introduction

Constant Maturity Swap (CMS) spread options pay a standard cap/floor pay-off based on the difference between two CMS rates. If the swap rates are denoted S_1 and S_2 and the strike K, the cap pay-off is

$$(S_1 - S_2 - K)^+$$

and the floor pay-off

$$(K - (S_1 - S_2))^+.$$

Usually, the swap underlying the first rate has longer tenor than the second one. Typical CMS spread options include 10Y-2Y, 30Y-2Y and 30Y-1Y.

This note describes a simple approach to price them. The CMS rates are modelled as correlated normally-distributed variables. This leads to an explicit pricing formula with the correlation being the calibration parameter.

Each of the two CMS rates used in the instrument can be priced using any CMS pricing mechanism. The approach used in the implementation is the pricing by replication described in OpenGamma Research [2011].

2. Notation

The two underlying swaps ($l = 1, 2$) have a common start date t_0 and fixed leg payment dates $(t_{l,i})_{1\leq i\leq n_l}$. The accrual fractions for each fixed period are $(\delta_{l,i})_{1\leq i\leq n_l}$. The floating leg payment dates are $(\tilde{t}_{l,i})_{1\leq i\leq \tilde{n}_l}$ and the fixing period start and end dates are $(s_{l,i})$ and $(e_{l,i})$.

The analysis framework is a *multi-curves setting*. There is one discounting curve denoted $P^D(s,t)$ and one forward curve $P^{j_l}(s,t)$ for each Ibor tenor where j_l is the relevant tenor. The floating leg of the two swaps potentially have different frequencies (e.g. a EUR CMS spread including a 1Y CMS).

The *annuities* are

$$A_{l,t} = \sum_{i=1}^{n_l} \delta_{l,i} P^D(t, t_{l,i}).$$

The swap rates are

$$S_{l,t} = \frac{\sum_{i=1}^{\tilde{n}_l} P^D(t, \tilde{t}_{l,i}) \left(\frac{P^{j_l}(t, s_{l,i})}{P^{j_l}(t, \tilde{e}_{l,i})} - 1 \right)}{A_{l,t}}.$$

As the CMS rate is fixed against an index, the underlying swap is always in practice a plain vanilla swap (with standard conventions).

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The CMS spread strike is denoted K, the payment date t_p and the coupon fixing date θ. The fixing date θ is related to the swap settlement date t_0 by the standard settlement lag. The notional is N and the payment accrual factor is α.

3. Modelling (implied CMS caplet ATM volatility)

The swap rates are martingales in their respective annuity numeraire but not necessarily in other numeraires. We chose to work with the $P^D(t, t_p)$ numeraire.

We work with a simplified normal assumption on the swap rate:

$$S_{l,\theta} = \mathrm{E}^{t_p}\left[S_{l,\theta}\right] + \sigma_l W_{l,\theta} \tag{1}$$

where $W_{l,t}$ is a Brownian motion in the $P^D(t, t_p)$ numeraire probability space. The correlation between the two Brownian motions W_l is ρ.

In practice there will be a correlation smile, as there is a volatility smile. For each CMS spread a different correlation $\rho(K)$ will be used in the pricing.

With the normal assumption, the price of a CMS coupon is

$$\mathrm{CMSCoupon}_l = P^D(0, t_p)\,\mathrm{E}^{t_p}\left[S_{l,\theta}\right] N\alpha$$

and the price of CMS caplet with strike L is

$$\mathrm{CMSCaplet}_{l,L} = P(0, t_p)\,\mathrm{E}^{t_p}\left[(S_{l,\theta} - L)^+\right] N\alpha = P^D(0, t_p)\mathrm{Bachelier}(\mathrm{E}^{t_p}\left[S_{l,\theta}\right], L, \sigma_l(L))N\alpha. \tag{2}$$

The Bachelier (normally distributed asset) pricing function is given by

$$\mathrm{Bachelier}(S, L, \sigma) = (S - L)N\left(\frac{S - L}{S\sigma\sqrt{\theta}}\right) + S\sigma\sqrt{\theta}\phi\left(\frac{S - L}{S\sigma\sqrt{\theta}}\right).$$

Note that two CMS prices are required for each rate: a coupon to obtain the forward $\mathrm{E}^{t_p}\left[S_{l,\theta}\right]$ and a cap to obtain the volatility σ_l. In this implementation, the caplet used for the implied volatility has a strike fixed at the ATM forward swap rate.

The volatility $\sigma(L)$ is the implied volatility obtained from the replication price, i.e.

$$\sigma_l(L) = \mathrm{Bachelier}^{-1}\left(\frac{\mathrm{CMSCaplet}_{l,L}}{P^D(0, t_p)N\alpha}\right).$$

In that simple approach the equation for the CMS spread is

$$S_{1,\theta} - S_{2,\theta} = \mathrm{E}^{t_p}\left[S_{1,\theta}\right] - \mathrm{E}^{t_p}\left[S_{2,\theta}\right] + \sigma W_\theta$$

where

$$\sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

and W_θ is a Brownian motion under the $P^D(0, t_p)$ numeraire.

The price of the CMS spread caplet is

$$\mathrm{CMSSpreadCaplet} = P^D(0, t_p)\mathrm{Bachelier}(\mathrm{E}^{t_p}\left[S_{1,\theta}\right] - \mathrm{E}^{t_p}\left[S_{2,\theta}\right], K, \sigma)N\alpha.$$

3.1. **Curve sensitivity.** The curve sensitivity is computed by using the derivative of composition on all the steps described above. The price of the CMS spreads is based on a calibration to CMS caps. The strike of those caps is set ATM forward. Those strikes can be considered to be constant and simply a parameter of the calibration, or themselves curve dependent. In this implementation it is considered that they are constant; the sensitivity of the price to the change of calibration strike is not computed. In that sense, the computed sensitivity is not equal to that that would be obtained by finite difference.

In the composition described above, one step requires a special attention; the implied volatility calibration to a CMSCaplet price. (the computation of implied volatility from Equation (2)). We don't have (and don't want to develop) the direct derivative of the equation solver. Instead, we rely on the implicit function theorem. For an equation of the type $f(x, y) = 0$ with a known solution (x_0, y_0), under some hypothesis, there exists around x_0 a implicit function $y = y(x)$ such that $f(x, y(x)) = 0$ and the derivative is given by

$$\frac{\partial}{\partial x} y(x_0) = -\left(\frac{\partial}{\partial y} f(x_0, y_0)\right)^{-1} \frac{\partial}{\partial x} f(x_0, y_0).$$

In this case, to compute the derivative of the implied volatility with respect to the strike (or caplet price, or expectation), we only require the derivative of the Bachelier function with respect to the strike and with respect to the volatility. Those figures are readily available in the implementation.

3.2. **SABR parameters sensitivity.** The SABR parameters sensitivity is computed similarly to the curve sensitivity using the derivative of composition. As for the curves sensitivity, an implicit function approach is used to compute the derivative with respect to the implied volatility.

4. Implementation

The CMS spread instruments are described in the class `CapFloorCMSSpreadDefinition` and `CapFloorCMSSpread`.

The pricing function under the normal hypothesis is implemented in the class `NormalPriceFunction`. The price and its derivatives with respect to the forward, the volatility and the strike are available. The inverse function, computing the volatility from the price is implemented in the class `NormalImpliedVolatilityFormula`.

The CMS spread cap/floor pricing is implemented in the class `CapFloorCMSSpreadSABRBinormalMethod`. The method contains a `presentValue`, a `presentValueCurveSensitivity` and an `impliedCorrelation` method.

References

OpenGamma Research. Replication pricing for linear and TEC format CMS. Analytics documentation, OpenGamma, April 2011.

Contents

Marc Henrard, Quantitative Research, OpenGamma
E-mail address: quant@opengamma.com